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Deriving Dualities in Pointfree Topology from Priestley Duality

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Abstract
There are several prominent duality results in pointfree topology. Hofmann–Lawson duality establishes that the category of continuous frames is dually equivalent to the category of locally compact sober spaces. This restricts to a dual equivalence between the categories of stably continuous frames and stably locally compact spaces, which further restricts to Isbell duality between the categories of compact regular frames and compact Hausdorff spaces. We show how to derive these dualities from Priestley duality for distributive lattices, thus shedding new light on these classic results.

Keywords Pointfree topology · Spatial frame · Continuous frame · Stably compact frame · Compact regular frame · Sober space · Locally compact space · Stably compact space · Compact Hausdorff space · Priestley duality

Mathematics Subject Classification 18F70 · 06D22 · 06D50 · 06E15 · 54D30 · 54D45

1 Introduction

In pointfree topology there is a well-known dual adjunction between the category Top of topological spaces and continuous maps and the category Frm of frames and frame homomorphisms (see, e.g., [15]). Let Sob be the full subcategory of Top consisting of sober spaces and SFrm the full subcategory of Frm consisting of spatial frames. The dual adjunction between Top and Frm then restricts to a dual equivalence between Sob and SFrm (see, e.g., [30, Sec. II–1]). Further restrictions yield the following classic results:

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- Hofmann–Lawson duality between the category ConFrm of continuous frames and proper frame homomorphisms and the category LKSob of locally compact sober spaces and proper maps [26].
- A dual equivalence between the full subcategory StCFrm of ConFrm consisting of stably continuous frames and the full subcategory StLKSp of LKSob consisting of stably locally compact spaces, which further restricts to a dual equivalence between the full subcategories StKFrm of stably compact frames and StKSp of stably compact spaces [5, 23, 29, 39].
- Isbell duality between the full subcategory KRFrm of Frm consisting of compact regular frames and the full subcategory KHaus of Top consisting of compact Hausdorff spaces [28].

Note that every frame homomorphism between compact regular frames is proper, and hence KRFrm is a full subcategory of StKFrm. Similarly, KHaus is a full subcategory of StKSp. We thus arrive at the diagram in Fig. 1, where a pair of squiggly arrows ($\substack{\leftsquigarrow \\ \rightsquigarrow}$) represents a dual adjunction and a squiggly left-right arrow ($\leftrightsquigarrow$) a dual equivalence. Also, $\mathsf{C} \leq \mathsf{D}$ stands for "C is a full subcategory of D" and $\mathsf{C} \preccurlyeq \mathsf{D}$ for "C is a non-full subcategory of D."

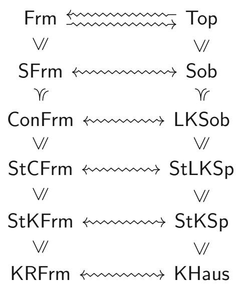
It is our aim to provide a different perspective on these dualities by utilizing Priestley duality [33, 34], which establishes a dual equivalence between the categories DLat of bounded distributive lattices and bounded lattice homomorphisms and Pries of Priestley spaces and Priestley morphisms.

It is well known (see, e.g., [30, Sec. II–3]) that DLat is equivalent to the category CohFrm of coherent frames, which is dually equivalent to the category Spec of spectral spaces. We recall that a stably compact frame L is *coherent* if compact elements join-generate L, and a stably compact space X is *spectral* if compact opens form a basis for X. Therefore, CohFrm is a full subcategory of StKFrm and Spec is a full subcategory of StKSp. By [13], Spec is isomorphic to Pries. Thus, Priestley duality can be derived from the dual equivalence of CohFrm and Spec.

On the other hand, since frames are special distributive lattices, they can be studied using the machinery of Priestley duality. This line of research was initiated by Pultr and Sichler [36] who showed that Priestley duality restricts to a dual equivalence between Frm and the category LPries of what we call L-spaces and L-morphisms (see Sect. 3). It was further developed in [3, 4, 9, 10, 37] where various properties of frames were characterized in the language of their Priestley duals. An alternative approach, using Spec instead of Pries, was investigated in [14, 38].

The exploration of Priestley spaces of frames has numerous applications, not only in pointfree topology, but in other areas as well. For example, nuclei play an important role in pointfree topology as they are kernels of frame homomorphisms (see, e.g., [32, p. 31]).

Fig. 1 Correspondence between various categories of frames and spaces



But they also arise in logic as they model the so-called lax modality [20, 24]. The resulting intuitionistic modal logic has various applications [1, 2, 19, 21, 25]. As was demonstrated in [11], nuclei also provide a unified semantic hierarchy for intuitionistic logic. It turns out that nuclei have a rather natural description in the language of Priestley spaces, which has resulted in numerous insights in understanding the complicated structure of the frame of nuclei of a given frame (see, e.g., [3, 4, 8, 10, 37]).

The goal of this paper is to continue the study of frames by means of their Priestley spaces. In particular, we provide dual descriptions of the categories SFrm, ConFrm, StCFrm, StKFrm, and KRFrm in the language of Priestley spaces. On the one hand, this yields an alternative proof of the dual equivalences mentioned at the beginning of the introduction, thus providing a new insight into these classic results in pointfree topology from the perspective of Priestley duality. On the other hand, it gives rise to new subcategories of Priestley spaces that are equivalent to such important categories of topological spaces as Sob, LKSob, StLKSp, StKSp, and KHaus. It is our belief that results of this nature can provide further insight and cross-fertilization between these beautiful branches of mathematics.

The paper is organized as follows. Section 2 introduces the categories of frames and spaces of interest, and presents the relevant dualities in more detail. Section 3 discusses Priestley duality and its restriction to frames. Section 4 characterizes spatial frames in the language of Priestley duality and connects the associated Priestley spaces with sober spaces. Section 5 further restricts this correspondence to continuous frames, their associated Priestley spaces, and locally compact sober spaces. This yields a new proof of Hofmann–Lawson duality. In Sect. 6 we derive the duality between stably continuous frames and stably locally compact spaces by describing stability in the language of Priestley spaces. This also gives a new proof of the duality between stably compact frames and stably compact spaces. Finally, Sect. 7 describes regularity in the language of Priestley spaces, thus providing an alternative proof of Isbell duality.

2 Frames and Spaces

We recall (see, e.g., [32, p. 10]) that a *frame* is a complete lattice L satisfying the join-infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for every $a \in L$ and $S \subseteq L$. A *frame homomorphism* is a map between frames that preserves finite meets and arbitrary joins. Let Frm be the category of frames and frame homomorphisms.

A filter F of a frame L is *completely prime* if $\bigvee S \in F$ implies $S \cap F \neq \varnothing$ and *Scott-open* if $\bigvee S \in F$ implies $\bigvee T \in F$ for some finite $T \subseteq S$. Clearly each completely prime filter is Scott-open. In fact, Scott-open filters are exactly the intersections of completely prime filters (see, e.g., [40, p. 101]).

A frame L is *spatial* if completely prime filters separate elements of L; that is, $a \not\leq b$ in L implies that there is a completely prime filter F with $a \in F$ and $b \notin F$. Equivalently, L is spatial iff Scott-open filters separate elements of L. It is well known (see, e.g., [32, p. 18]) that L is spatial iff L is isomorphic to the frame $\mathcal{O}(X)$ of open sets of a topological space X (hence the name). Let SFrm be the full subcategory of Frm consisting of spatial frames.

There are two relations on frames that are of particular importance to us. We recall that if L is a frame and $a \in L$, then the *pseudocomplement* of a is $a^* = \bigvee \{x \in L \mid a \wedge x = 0\}$.

Definition 2.1 Let L be a frame and $a, b \in L$.

(1) (see, e.g., [22, p. 49]) We say that a is *way below* b and write $a \ll b$ if for each $S \subseteq L$, from $b \leq \bigvee S$ it follows that $a \leq \bigvee T$ for some finite $T \subseteq S$.
(2) (see, e.g., [30, p. 80]) We say that a is *well inside* b and write $a \prec b$ if $a^* \vee b = 1$.

A frame L is *continuous* if

$$a = \bigvee \{b \in L \mid b \ll a\}$$

and L is *regular* if

$$a = \bigvee \{b \in L \mid b \prec a\}$$

for all $a \in L$.

Each frame homomorphism $h : L \to M$ preserves $\prec$ (that is, $a \prec b$ implies $h(a) \prec h(b)$), but may not preserve $\ll$. We call h *proper* provided h preserves $\ll$ (that is, $a \ll b$ implies $h(a) \ll h(b)$).

Definition 2.2 Let ConFrm be the category of continuous frames and proper frame homomorphisms between them.

We call an element a of a frame L *compact* if $a \ll a$ and the frame L *compact* if the top element 1 is compact. We also call $\ll$ *stable* if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$.

Definition 2.3 (1) (see, e.g., [22, p. 488]) A frame L is *stably continuous* if L is continuous and $\ll$ is stable. Let StCFrm be the full subcategory of ConFrm consisting of stably continuous frames.
(2) (see, e.g., [22, p. 488]) A frame L is *stably compact* if L is compact and stably continuous. Let StKFrm be the full subcategory of StCFrm consisting of stably compact frames.
(3) (see, e.g., [32, p. 133]) Let KRFrm be the full subcategory of Frm consisting of compact regular frames.

We note that if L is compact, then $a \prec b$ implies $a \ll b$, and if L is regular, then $a \ll b$ implies $a \prec b$ (see, e.g., [32, Lem. VII–5.2.1]). Therefore, if L is compact regular, then the way below and well inside relations on L coincide. Thus, since $a \prec b, c$ implies $a \prec b \wedge c$, every compact regular frame is stably compact. Consequently, KRFrm is a full subcategory of StKFrm.

Table 1 contains the categories of frames that we will be concerned with in this paper. We next turn our attention to the categories of spaces that correspond to the categories of frames in Table 1. The definitions that follow are well known; see, e.g., [22].

For a partially ordered set P and $S \subseteq P$, we write

$${\uparrow}S = \{x \in P \mid s \leq x \text{ for some } s \in S\} \quad \text{and} \quad {\downarrow}S = \{x \in P \mid x \leq s \text{ for some } s \in S\}.$$

Then S is an *upset* if $S = {\uparrow}S$, S is a *downset* if $S = {\downarrow}S$, and S is a *biset* if it is both an upset and a downset. For a singleton $S = \{x\}$ we write ${\uparrow}x$ for ${\uparrow}S$ and ${\downarrow}x$ for ${\downarrow}S$.

Let X be a topological space. We recall that a closed subset A of X is *irreducible* if A cannot be written as a union of two closed proper subsets, and that X is *sober* if each irreducible subset of X is the closure of a unique point in X. In particular, every sober space is T_0.

The space X is *locally compact* if for each open set U and $x \in U$ there is an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$. A subset of X is *saturated* if it is an

Table 1 Categories of frames

Category	Objects	Morphisms
Frm	Frames	Frame homomorphisms
SFrm	Spatial frames	Frame homomorphisms
ConFrm	Continuous frames	Proper frame homomorphisms
StCFrm	Stably continuous frames	Proper frame homomorphisms
StKFrm	Stably compact frames	Proper frame homomorphisms
KRFrm	Compact regular frames	Frame homomorphisms

intersection of open sets, and X is *coherent* if the intersection of two compact saturated sets is again compact.

The *specialization preorder* $\leq$ on X is defined by $x \leq y$ iff $x \in \mathsf{cl}(\{y\})$, where cl is topological closure. Observe that saturated sets of X are exactly upsets in the specialization preorder, which is a partial order iff X is a T_0-space.

Definition 2.4 (1) Let Top be the category of topological spaces and continuous maps, and let Sob be the full subcategory of Top consisting of sober spaces.
(2) A continuous map $f : X \to Y$ between topological spaces is *proper* if

(i) $\downarrow f(A)$ is closed for each closed set $A \subseteq X$, where $\downarrow$ is the downset in the specialization preorder on X.
(ii) $f^{-1}(B)$ is compact for each compact saturated set $B \subseteq Y$.

(3) Let LKSob be the category of locally compact sober spaces and proper maps between them.
(4) We call X *stably locally compact* if X is locally compact, sober, and coherent. Let StLKSp be the full subcategory of LKSob consisting of stably locally compact spaces.
(5) We call X *stably compact* if X is compact and stably locally compact. Let StKSp be the full subcategory of StLKSp consisting of stably compact spaces.
(6) Let KHaus be the full subcategory of Sob consisting of compact Hausdorff spaces.

Remark 2.5 (1) By [22, Lem. VI–6.21], if X is sober and Y is locally compact, then (i) follows from (ii) in Definition 2.4(2).
(2) In compact Hausdorff spaces, the specialization order is the identity. Hence, compact saturated sets are simply closed sets. Therefore, since every compact Hausdorff space is sober and locally compact, KHaus is a full subcategory of StKSp.

Table 2 contains the categories of topological spaces that we are interested in. There is a well-known dual adjunction $(\mathcal{O}, pt)$ between Top and Frm (see, e.g., [15]). The contravariant functors $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}$ and $pt : \mathsf{Frm} \to \mathsf{Top}$ are constructed as follows. The functor $\mathcal{O}$ maps a topological space X to its frame of open sets, and a continuous map $f : X \to Y$ to $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. The functor pt maps a frame L to its space of points, where a *point* is a completely prime filter of L. The topology on $pt(L)$ is the range of the map $\zeta : L \to \wp(pt(L))$ given by $\zeta(a) = \{x \in pt(L) \mid a \in x\}$, where $\wp(X)$ denotes the powerset of a set X. A frame homomorphism $h : L \to M$ is mapped to $h^{-1} : pt(M) \to pt(L)$.

Restricting the range of these functors yields the following well-known duality results:

Theorem 2.6 (1) SFrm *is dually equivalent to* Sob.
(2) (Hofmann–Lawson duality) ConFrm *is dually equivalent to* LKSob.

Table 2 Categories of spaces

Category	Objects	Morphisms
Top	Topological spaces	Continuous maps
Sob	Sober spaces	Continuous maps
LKSob	Locally compact sober spaces	Proper maps
StLKSp	Stably locally compact spaces	Proper maps
StKSp	Stably compact spaces	Proper maps
KHaus	Compact Hausdorff spaces	Continuous maps

(3) StCFrm *is dually equivalent to* StLKSp.
(4) StKFrm *is dually equivalent to* StKSp.
(5) (Isbell duality) KRFrm *is dually equivalent to* KHaus.

For Theorem 2.6(1) see, e.g., [30, Sec. II–1]. Hofmann–Lawson duality was established in [26] (see also [22, Prop. V–5.20]). Theorems 2.6(3) and 2.6(4) go back to [5, 23, 29, 39] (see also [22, Thm. VI–7.4]). Isbell duality was established in [28] (see also [6] and [30, Sec. VII–4]). We thus obtain the diagram in Fig. 1.

3 Priestley Duality for Frames

As is customary, we call a subset of a topological space X *clopen* if it is both closed and open. Then X is *zero-dimensional* if it has a basis of clopen sets. *Stone spaces* are zero-dimensional compact Hausdorff spaces.

Definition 3.1 A *Priestley space* is a pair $(X, \leq)$ where X is a compact space and $\leq$ is a partial order on X satisfying the *Priestley separation axiom*:

$$\text{If } x \not\leq y, \text{ then there is a clopen upset } U \text{ with } x \in U \text{ and } y \notin U.$$

For a Priestley space $(X, \leq)$ we simply write X and note that every Priestley space is a Stone space.

A *Priestley morphism* is a continuous map $f : X \to Y$ between Priestley spaces that is order-preserving. Let Pries be the category of Priestley spaces and Priestley morphisms. Let also DLat by the category of bounded distributive lattices and bounded lattice homomorphisms.

Theorem 3.2 (Priestley duality) Pries *is dually equivalent to* DLat.

Remark 3.3 (1) For a bounded distributive lattice D, the *Priestley space* X_D of D is given by the set of prime filters of D ordered by inclusion and topologized by the subbasis

$$\{\varphi(a) \mid a \in D\} \cup \{\varphi(a)^c \mid a \in D\},$$

where $\varphi : D \to \wp(X_D)$ is the Stone map given by $\varphi(a) = \{x \in X_D \mid a \in x\}$ for each $a \in D$.

(2) The contravariant functors $\mathscr{X}$: DLat $\to$ Pries and $\mathscr{D}$: Pries $\to$ DLat establishing Priestley duality are described as follows. The functor $\mathscr{X}$ sends a bounded distributive lattice D to the Priestley space X_D of D and a bounded lattice homomorphism $h : D \to E$ to the Priestley morphism $h^{-1} : X_E \to X_D$. The functor $\mathscr{D}$ sends a Priestley space X

to the bounded distributive lattice $\mathsf{ClopUp}(X)$ of clopen upsets of X and a Priestley morphism $f : X \to Y$ to the bounded lattice homomorphism $f^{-1} : \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$. The natural isomorphisms are given by $\varphi : D \to \mathscr{D}\mathscr{X}D$ defined above and $\varepsilon : X \to \mathscr{X}\mathscr{D}X$ defined by $\varepsilon(x) = \{U \in \mathsf{ClopUp}(X) \mid x \in U\}$ for each $x \in X$.

Since frames are special bounded distributive lattices, we can restrict Priestley duality to obtain a category that is dual to Frm. It is well known that a bounded distributive lattice is a frame iff it is a complete Heyting algebra (see, e.g., [18, Prop. 1.5.4]). Therefore, we can describe the dual category of Frm using Esakia duality [17]. We recall that a Priestley space is an *Esakia space* if U clopen implies that $\downarrow U$ is clopen. The following are well-known properties of Priestley and Esakia spaces that we will use frequently. For a subset F of a poset X, we write $\min(F)$ and $\max(F)$ for the sets of minimal and maximal elements of F, respectively.

Lemma 3.4 (see, e.g., [18, 35]) *Let X be a Priestley space.*

(1) *The collection of clopen upsets and clopen downsets of X forms a subbasis for X.*
(2) *Every open upset/downset is a union of clopen upsets/downsets.*
(3) *Every closed upset/downset is an intersection of clopen upsets/downsets.*
(4) *If $F \subseteq X$ is closed, then both $\uparrow F$ and $\downarrow F$ are closed.*
(5) *If F is a closed set, then $\min(F)$ and $\max(F)$ are nonempty. In fact, for every $x \in F$ there exist $y \in \min(F)$ and $z \in \max(F)$ such that $y \leq x \leq z$. Consequently, if F is a closed upset, then $F = \uparrow\min(F)$ and if F is a closed downset, then $F = \downarrow\max(F)$.*
(6) *If $\mathcal{P}$ is a prime filter of $\mathsf{ClopUp}(X)$, then $\bigcap\mathcal{P} = \uparrow x$ for a unique $x \in X$.*

Suppose additionally that X is an Esakia space.

(7) $\mathsf{cl}\uparrow F = \uparrow\mathsf{cl}\, F$. *Consequently, the closure of an upset is an upset and the interior of a downset is a downset.*

Let $D \in \mathsf{DLat}$ and X_D be its Priestley space. Then D is a Heyting algebra iff X_D is an Esakia space [17], and D is a complete Heyting algebra iff X_D is an extremally order-disconnected Esakia space [7, Thm. 2.4(2)], where we recall that an Esakia space is *extremally order-disconnected* if $\mathsf{cl}\, U$ is open for every open upset U. We thus arrive at the following well-known result. It was first proved in [36, Thm. 2.3] without using Esakia duality. For the formulation below, see [3, Thm. 3.7].

Theorem 3.5 *Let D be a bounded distributive lattice and X_D its Priestley space. Then D is a frame iff X_D is an extremally order-disconnected Esakia space.*

The following well-known fact (see, e.g., [7, Lem. 2.3]) will be used throughout.

Lemma 3.6 *For a frame L, its Priestley space X_L, and $S \subseteq L$, we have*

$$\varphi\left(\bigvee S\right) = \mathsf{cl}\left(\bigcup\{\varphi(s) \mid s \in S\}\right).$$

Frame homomorphisms are dually characterized by Priestley morphisms that satisfy the following additional condition.

Lemma 3.7 ([36, Sec. 2.5]) *Let $L, M \in \mathsf{Frm}$, $h : L \to M$ be a bounded lattice homomorphism, and $f = \mathscr{X}(h)$. Then h is a frame homomorphism iff $f^{-1}\,\mathsf{cl}\, U = \mathsf{cl}\, f^{-1}(U)$ for all open upsets U of X_L.*

Since frames are also known as locales (see, e.g., [32]), we introduce the following terminology.

Definition 3.8 (L-spaces)

(1) A *localic space* or simply an *L-space* is an extremally order-disconnected Esakia space.
(2) An *L-morphism* is a Priestley morphism $f : X \to Y$ between L-spaces such that $f^{-1}\operatorname{cl} U = \operatorname{cl} f^{-1}(U)$ for all open upsets U of Y.
(3) Let LPries be the category of L-spaces and L-morphisms.

Theorem 3.9 (Pultr–Sichler [36, Cor. 2.5]) Frm *is dually equivalent to* LPries.

Remark 3.10 The functors establishing Pultr–Sichler duality are the restrictions of the functors $\mathscr{X} : \mathsf{DLat} \to \mathsf{Pries}$ and $\mathscr{D} : \mathsf{Pries} \to \mathsf{DLat}$ establishing Priestley duality, and the units of this duality are the restrictions of the units φ and ε of Priestley duality (see Remark 3.3(2)).

4 Priestley Duality for Spatial Frames

Let L be a frame and X_L the Priestley space of L. Since completely prime filters are prime filters, $pt(L)$ is a subset of X_L, which from now on will be denoted by Y_L. In [37] elements of Y_L are called *L-points* and in [3] they are called *nuclear points*. We follow the terminology of [38] and call them *localic points*. In addition, we refer to Y_L as the *localic part* of X_L. The next lemma shows that open subsets of Y_L are exactly the intersections of clopen upsets of X_L with Y_L.

Lemma 4.1 [3, Lem. 5.3(1)] *Let L be a frame, X_L its Priestley space, and $Y_L \subseteq X_L$ the localic part of X_L. Then $\zeta(a) = \varphi(a) \cap Y_L$ for each $a \in L$.*

The following characterization of Y_L was given in [37, Prop. 2.9]; see also [9, Lem. 5.1].

Lemma 4.2 *Let L be a frame, X_L its Priestley space, $Y_L \subseteq X_L$ the localic part of X_L, and $x \in X_L$. Then $x \in Y_L$ iff ${\downarrow}x$ is clopen.*

This motivates the following definition.

Definition 4.3 Let X be an L-space. We call

$$Y := \{y \in X \mid {\downarrow}y \text{ is clopen}\}$$

the *localic part of* X. We view Y as a topological space, where $U \subseteq Y$ is open iff $U = V \cap Y$ for some $V \in \mathsf{ClopUp}(X)$.

Definition 4.4 Let X be an L-space and Y the localic part of X. We call a closed upset F of X a *Scott upset* if $\min(F) \subseteq Y$.

Scott upsets were introduced in [12] where it was shown that they provide a characterization of Scott-open filters of a frame in the language of Priestley spaces. The next lemma provides a characterization of Scott upsets.

Lemma 4.5 ([12, Lem. 5.1]) *Let X be an L-space and F a closed upset of X. Then F is a Scott upset iff for every open upset U of X, from $F \subseteq \operatorname{cl} U$ it follows that $F \subseteq U$.*

Remark 4.6 In [37] closed sets satisfying the property in Lemma 4.5 are called *L-compact sets*. Thus, Scott upsets are exactly the upsets of L-compact sets.

Spatial frames are characterized by the following theorem. The equivalence (1)⇔(2) is proved in [3, Thm. 5.5] and the equivalence (1)⇔(3) in [37, Sec. 2.11].

Theorem 4.7 *Let L be a frame, X_L its Priestley space, and $Y_L \subseteq X_L$ the localic part of X_L. The following are equivalent.*

(1) *L is spatial.*
(2) *Y_L is dense in X_L.*
(3) *For clopen upsets U, V of X_L, from $U \not\subseteq V$ it follows that there is a Scott upset F of X_L such that $F \subseteq U$ but $F \not\subseteq V$.*

Remark 4.8 By Theorem 4.7(2), if U is clopen in X_L, then $\mathsf{cl}(U \cap Y_L) = U$. In particular, for $a \in L$, by Lemma 4.1 we have $\mathsf{cl}\,\zeta(a) = \mathsf{cl}(\varphi(a) \cap Y_L) = \varphi(a)$ (see also [37, Sec. 2.12]).

Definition 4.9 (1) An L-space X is *L-spatial* or simply an *SL-space* if the localic part Y of X is dense in X.
(2) Let SLPries be the full subcategory of LPries consisting of SL-spaces.

As a consequence of Theorems 3.9 and 4.7 we obtain:

Corollary 4.10 SFrm *is dually equivalent to* SLPries.

We next connect SLPries with Sob. In order to do so, we show that mapping an L-space to its localic part is functorial. For this we need the following lemmas.

Lemma 4.11 *Let X be an L-space and Y the localic part of X. Then Y is a sober space.*

Proof By Lemmas 4.1 and 4.2, Y is homeomorphic to $pt(\mathsf{ClopUp}(X))$. Thus, Y is sober (see, e.g., [32, p. 20]). □

Lemma 4.12 *Let X_1, X_2 be L-spaces, Y_1, Y_2 their localic parts, and $f : X_1 \to X_2$ an L-morphism.*

(1) *$f(Y_1) \subseteq Y_2$.*
(2) *The restriction $f : Y_1 \to Y_2$ is a well-defined continuous map.*

Proof (1) Let $y \in Y_1$ and set $U = ({\downarrow} f(y))^c$. Since $y \notin f^{-1}(U)$ and $f^{-1}(U)$ is an upset, ${\downarrow} y \cap f^{-1}(U) = \varnothing$. Because $y \in Y_1$, we have ${\downarrow} y$ is open, so $y \notin \mathsf{cl}\, f^{-1}(U) = f^{-1}(\mathsf{cl}\, U)$ (see Definition 3.8(2)). Therefore, $f(y) \notin \mathsf{cl}\, U$, and so $f(y) \in \mathsf{int}\, {\downarrow} f(y)$. By Lemma 3.4(7), $\mathsf{int}\, {\downarrow} f(y)$ is a downset. Thus, $f(y) \in \mathsf{int}\, {\downarrow} f(y)$ implies that ${\downarrow} f(y) = \mathsf{int}\, {\downarrow} f(y)$, hence ${\downarrow} f(y)$ is open. Consequently, $f(y) \in Y_2$.

(2) That the restriction of f is well defined follows from (1). For continuity, it suffices to show that $f^{-1}(U \cap Y_2) \cap Y_1$ is open in Y_1 for every clopen upset U of X_2. By (1), $f^{-1}(U \cap Y_2) \cap Y_1 = f^{-1}(U) \cap Y_1$. Since f is a Priestley morphism, $f^{-1}(U)$ is a clopen upset of X_1. Thus, $f^{-1}(U) \cap Y_1$ is an open subset of Y_1. □

We define a functor $\mathscr{Y}$: SLPries → Sob by sending an SL-space X to its localic part Y, and an L-morphism $f : X_1 \to X_2$ to its restriction $f : Y_1 \to Y_2$. It follows easily from Lemmas 4.11 and 4.12(2) that $\mathscr{Y}$ is a well-defined covariant functor.

Theorem 4.13 *$\mathscr{Y}$ is essentially surjective.*

Proof Suppose Z is a sober space. Then Z is homeomorphic to $pt(\mathcal{O}(Z))$ (see, e.g., [32, p. 20]). Let X be the Priestley space of $\mathcal{O}(Z)$. It follows from Lemma 4.1 that $pt(\mathcal{O}(Z))$ is (homeomorphic to) the localic part of X. □

To show that $\mathscr{Y}$ is full and faithful, we need the following lemmas.

Lemma 4.14 *Let X be an L-space and Y the localic part of X.*

(1) $\mathsf{cl}\, U \cap Y = U \cap Y$ *for each open upset U of X.*
(2) $\mathsf{cl}\, U \cap Y = U$ *for each open set U of Y.*

Proof (1) We clearly have that $U \cap Y \subseteq \mathsf{cl}\, U \cap Y$. For the reverse inclusion, let $y \in \mathsf{cl}\, U \cap Y$. Since $y \in Y$, we have that ${\downarrow} y$ is open. Hence, ${\downarrow} y \cap U \neq \varnothing$. Thus, there is $x \in U$ with $x \leq y$. Since U is an upset, we must have $y \in U$, so $y \in U \cap Y$.

(2) Since U is open in Y, there is a clopen upset V of X such that $U = V \cap Y$. Thus, $\mathsf{cl}\, U \subseteq V$, and hence $U \subseteq \mathsf{cl}\, U \cap Y \subseteq V \cap Y = U$. □

Lemma 4.15 *Let X be an SL-space and Y the localic part of X. For open subsets U and V of Y we have* $\mathsf{cl}\, U \cap \mathsf{cl}\, V = \mathsf{cl}(U \cap V)$.

Proof Let U and V be open subsets of Y. Then there exist $U', V' \in \mathsf{ClopUp}(X)$ such that $U = U' \cap Y$ and $V = V' \cap Y$. Since X is L-spatial, Y is dense in X, so $U' = \mathsf{cl}\, U$ and $V' = \mathsf{cl}\, V$. Therefore, because $U' \cap V'$ is clopen in X, we have

$$U' \cap V' = \mathsf{cl}((U' \cap V') \cap Y) = \mathsf{cl}((U' \cap Y) \cap (V' \cap Y)) = \mathsf{cl}(U \cap V).$$

Thus, $\mathsf{cl}\, U \cap \mathsf{cl}\, V = U' \cap V' = \mathsf{cl}(U \cap V)$. □

Lemma 4.16 *Let X_1, X_2 be SL-spaces, Y_1, Y_2 their localic parts, $g : Y_1 \to Y_2$ a continuous map, and $x \in X_1$. Then $\mathcal{P}_x := \{U \in \mathsf{ClopUp}(X_2) \mid x \in \mathsf{cl}[g^{-1}(U \cap Y_2)]\}$ is a prime filter in* $\mathsf{ClopUp}(X_2)$.

Proof It is easy to see that $\mathcal{P}_x$ is an upset and that $U \cup V \in \mathcal{P}_x$ implies $U \in \mathcal{P}_x$ or $V \in \mathcal{P}_x$. Let $U, V \in \mathcal{P}_x$. Then $x \in \mathsf{cl}[g^{-1}(U \cap Y_2)], \mathsf{cl}[g^{-1}(V \cap Y_2)]$. Since $U \cap Y_2$, $V \cap Y_2$ are open in Y_2 and g is continuous, $g^{-1}(U \cap Y_2)$, $g^{-1}(V \cap Y_2)$ are open in Y_1. Therefore, by Lemma 4.15,

$$\begin{aligned} x \in \mathsf{cl}[g^{-1}(U \cap Y_2)] \cap \mathsf{cl}[g^{-1}(V \cap Y_2)] &= \mathsf{cl}[g^{-1}(U \cap Y_2) \cap g^{-1}(V \cap Y_2)] \\ &= \mathsf{cl}[g^{-1}((U \cap V) \cap Y_2)]. \end{aligned}$$

Thus, $U \cap V \in \mathcal{P}_x$, and hence $\mathcal{P}_x$ is a prime filter. □

Lemma 4.17 *Suppose that X_1, X_2 are SL-spaces, Y_1, Y_2 are their localic parts, and $g : Y_1 \to Y_2$ is a continuous map. Then there is an L-morphism $f : X_1 \to X_2$ which extends g.*

Proof Let $x \in X_1$. By Lemma 4.16, $\mathcal{P}_x = \{U \in \mathsf{ClopUp}(X_2) \mid x \in \mathsf{cl}[g^{-1}(U \cap Y_2)]\}$ is a prime filter of $\mathsf{ClopUp}(X)$. By Lemma 3.4(6), $\bigcap \mathcal{P}_x = {\uparrow} z$ for a unique $z \in X_2$. Define $f : X_1 \to X_2$ by $f(x) = z$ for each $x \in X_1$. It is clear that f is a well-defined map. To see that f extends g, suppose $y \in Y_1$. Then

$$\begin{aligned} {\uparrow} g(y) &= \bigcap \{U \in \mathsf{ClopUp}(X_2) \mid g(y) \in U\} \\ &= \bigcap \{U \in \mathsf{ClopUp}(X_2) \mid g(y) \in U \cap Y_2\} \end{aligned}$$

$$= \bigcap\{U \in \mathsf{ClopUp}(X_2) \mid y \in g^{-1}(U \cap Y_2)\}$$
$$= \bigcap\{U \in \mathsf{ClopUp}(X_2) \mid y \in \mathsf{cl}[g^{-1}(U \cap Y_2)]\} = \bigcap \mathcal{P}_y,$$

where the second to last equality follows from Lemma 4.14(2). Thus, $f(y) = g(y)$ by definition of f.

To see that f is continuous, suppose U is a clopen upset of X_2. Then $U \cap Y_2$ is an open subset of Y_2. Since g is continuous, $g^{-1}(U \cap Y_2)$ is an open subset of Y_1, and hence $\mathsf{cl}\, g^{-1}(U \cap Y_2)$ is a clopen upset of X_1 (because X_1 is L-spatial). But $\mathsf{cl}\, g^{-1}(U \cap Y_2) = f^{-1}(U)$ since by definition of f we have

$$x \in \mathsf{cl}\, g^{-1}(U \cap Y_2) \iff U \in \mathcal{P}_x \iff \bigcap \mathcal{P}_x \subseteq U \iff {\uparrow} f(x) \subseteq U$$
$$\iff f(x) \in U \iff x \in f^{-1}(U),$$

where in the second equivalence we use that U is clopen, hence compact. Thus, $f^{-1}(U)$ is a clopen upset of X_1. Since clopen downsets are complements of clopen upsets, we also obtain $f^{-1}(D)$ is a clopen downset for each clopen downset D of X_2. Thus, f is continuous since clopen upsets and clopen downsets form a subbasis of X_2 (see Lemma 3.4(1)).

To see that f is order-preserving, since $\mathsf{cl}\, g^{-1}(U \cap Y_2) = f^{-1}(U)$ is an upset, $x \leq z$ implies $\mathcal{P}_x \subseteq \mathcal{P}_z$. Therefore, $\bigcap \mathcal{P}_z \subseteq \bigcap \mathcal{P}_x$, and hence $f(x) \leq f(z)$. Thus, f is order-preserving.

It is left to prove that $\mathsf{cl}\, f^{-1}(U) = f^{-1}\, \mathsf{cl}\, U$ for each open upset U of X_2. The left-to-right inclusion follows from the continuity of f. For the right-to-left inclusion, let $x \in f^{-1}(\mathsf{cl}\, U)$. Then $f(x) \in \mathsf{cl}\, U$, so ${\uparrow} f(x) \subseteq \mathsf{cl}\, U$ by Lemma 3.4(7). Therefore, $\bigcap \mathcal{P}_x \subseteq \mathsf{cl}\, U$ and $\mathsf{cl}\, U$ is open since U is an open upset and X_2 is an L-space. Since $\mathcal{P}_x$ is a filter, by compactness there is $V \in \mathcal{P}_x$ such that $V \subseteq \mathsf{cl}\, U$. The former means that $x \in \mathsf{cl}\, g^{-1}(V \cap Y_2) = \mathsf{cl}\, f^{-1}(V \cap Y_2)$, which together with the latter and Lemma 4.14(1) gives

$$x \in \mathsf{cl}\, f^{-1}(V \cap Y_2) \subseteq \mathsf{cl}(f^{-1}(\mathsf{cl}\, U \cap Y_2)) = \mathsf{cl}\, f^{-1}(U \cap Y_2) \subseteq \mathsf{cl}\, f^{-1}(U).$$

Thus, f is an L-morphism. □

Theorem 4.18 *$\mathscr{Y}$ is full and faithful.*

Proof To see that $\mathscr{Y}$ is full, suppose $g : Y_1 \to Y_2$ is a continuous map. By Lemma 4.17, there is an L-morphism $f : X_1 \to X_2$ extending g. Thus, $\mathscr{Y} f = g$. To see that $\mathscr{Y}$ is faithful, suppose $f_1, f_2 : X_1 \to X_2$ are L-morphisms with $f_1 \neq f_2$. Since Y_1 is a dense subset of X_1 and X_2 is Hausdorff, f_1 and f_2 must be the unique extensions of their restrictions $\mathscr{Y} f_1$ and $\mathscr{Y} f_2$ to Y_1 (see, e.g., [16, p. 70]). Thus, $\mathscr{Y} f_1 \neq \mathscr{Y} f_2$. □

Corollary 4.19 SLPries *is equivalent to* Sob.

Proof By Theorems 4.13 and 4.18, $\mathscr{Y}$ is essentially surjective, full, and faithful. Thus, $\mathscr{Y}$ is an equivalence (see, e.g., [31, p. 93]). □

Combining Corollaries 4.10 and 4.19 yields an alternative proof of the well-known result mentioned in the introduction that SFrm is dually equivalent to Sob. In the next section we will restrict the correspondence between SFrm, SLPries, and Sob to obtain an alternative proof of Hofmann–Lawson duality.

5 Deriving Hofmann–Lawson Duality

Definition 5.1 Suppose X is an L-space.

(1) For $U, V \in \mathsf{ClopUp}(X)$, define $V \ll U$ provided for each open upset W of X we have $U \subseteq \mathsf{cl}\, W$ implies $V \subseteq W$.
(2) For $U \in \mathsf{ClopUp}(X)$, define the *kernel of U* as

$$\ker U = \bigcup\{V \in \mathsf{ClopUp}(X) \mid V \ll U\}.$$

If X is the Priestley dual of a frame L and $U = \varphi(a)$ for some $a \in L$, we simply write $\ker(a)$ for $\ker U$.

Lemma 5.2 *Let X be an L-space and U, V clopen upsets of X.*

(1) $\ker U$ *is an open upset contained in* U.
(2) $\ker$ *is monotone.*
(3) $V \subseteq \ker U$ *iff* $V \ll U$.
(4) $U \subseteq \mathsf{cl}\, W$ *implies* $\ker U \subseteq W$ *for each open upset* W.

Moreover, if X is the Priestley space of a frame L and $a, b \in L$, then

(5) $a \ll b$ *iff* $\varphi(a) \ll \varphi(b)$ *iff* $\varphi(a) \subseteq \ker(b)$.

If in addition L is spatial, then

(6) $a \ll b$ *iff* $\varphi(a) \subseteq {\uparrow}(\varphi(b) \cap Y_L)$.

Proof (1) This is immediate from the definition of $\ker U$ since $V \ll U$ implies $V \subseteq U$.

(2) Let $U_1 \subseteq U_2$, and let V be a clopen upset with $V \ll U_1$. Suppose W is an open upset such that $U_2 \subseteq \mathsf{cl}\, W$. Then $U_1 \subseteq \mathsf{cl}\, W$, so $V \subseteq W$. Hence, $V \ll U_2$. Consequently, $\ker U_1 \subseteq \ker U_2$.

(3) The right-to-left implication is immediate from the definition. For the left-to-right implication, if $V \subseteq \ker U$ then by compactness there is a clopen upset $V' \ll U$ such that $V \subseteq V'$. Therefore, $V \ll U$.

(4) Suppose $U \subseteq \mathsf{cl}\, W$ and let $x \in \ker U$. Then there is a clopen upset V of X with $x \in V \ll U$. Hence, $x \in V \subseteq W$.

(5) Suppose that $a \ll b$ and U is an open upset of X such that $\varphi(b) \subseteq \mathsf{cl}\, U$. Since $U = \bigcup \varphi[S]$ for some $S \subseteq L$, by Lemma 3.6, we have

$$\varphi(b) \subseteq \mathsf{cl}\left(\bigcup \varphi[S]\right) = \varphi\left(\bigvee S\right).$$

Therefore, $b \le \bigvee S$. Since $a \ll b$, there is a finite $T \subseteq S$ such that $a \le \bigvee T$. Thus,

$$\varphi(a) \subseteq \varphi\left(\bigvee T\right) = \bigcup \varphi[T] \subseteq \bigcup \varphi[S] = U.$$

Consequently, $\varphi(a) \ll \varphi(b)$.

Conversely, suppose that $\varphi(a) \ll \varphi(b)$. Therefore, $\varphi(b) \subseteq \mathsf{cl}\, U$ implies $\varphi(a) \subseteq U$ for all open upsets $U \subseteq X_L$. Let $b \le \bigvee S$ for some $S \subseteq L$. Then

$$\varphi(b) \subseteq \varphi\left(\bigvee S\right) = \mathsf{cl} \bigcup \varphi[S].$$

By assumption, $\varphi(a) \subseteq \bigcup \varphi[S]$. Since $\varphi(a)$ is compact, $\varphi(a) \subseteq \varphi[T] = \varphi(\bigvee T)$ for some finite $T \subseteq S$. Thus, $a \le \bigvee T$, and hence $a \ll b$.

This proves that $a \ll b$ iff $\varphi(a) \ll \varphi(b)$. The latter is equivalent to $\varphi(a) \subseteq \ker(b)$ by (3).

(6) Suppose $\varphi(a) \nsubseteq \uparrow(\varphi(b) \cap Y_L)$. Then there is $x \in \varphi(a)$ such that $\downarrow x \cap \varphi(b) \cap Y_L = \varnothing$. Therefore, $\varphi(b) \cap Y_L \subseteq (\downarrow x)^c$. Since L is spatial, Remark 4.8 implies that $\varphi(b) \subseteq \mathsf{cl}(\downarrow x)^c$. Hence, $\varphi(b)$ is contained in the closure of the open upset $(\downarrow x)^c$, while $\varphi(a) \nsubseteq (\downarrow x)^c$. Thus, $a \not\ll b$ by (5).

For the converse, suppose that $\varphi(a) \subseteq \uparrow(\varphi(b) \cap Y_L)$ and $b \le \bigvee S$ for some $S \subseteq L$. Then $\varphi(b) \subseteq \mathsf{cl} \bigcup \varphi[S]$. Therefore, $\varphi(b) \cap Y_L \subseteq \mathsf{cl}(\bigcup \varphi[S]) \cap Y_L = \bigcup \varphi[S] \cap Y_L \subseteq \bigcup \varphi[S]$ by Lemma 4.14(1). Thus, $\uparrow(\varphi(b) \cap Y_L) \subseteq \bigcup \varphi[S]$. By assumption, $\varphi(a) \subseteq \bigcup \varphi[S]$. Since $\varphi(a)$ is compact, $\varphi(a) \subseteq \bigcup \varphi[T]$ for some finite $T \subseteq S$. Hence, $a \le \bigvee T$, and so $a \ll b$. □

Remark 5.3 The equivalence of the first two items of Lemma 5.2(5) was first proved in [36, Prop. 3.6].

Definition 5.4 Let X be an L-space.

(1) We call a clopen upset U of X *packed* if $\ker U$ is dense in U.
(2) We call X a *continuous L-space* or simply a *CL-space* if each clopen upset of X is packed.

Theorem 5.5 *Let L be a frame, X_L its Priestley space, and $a \in L$.*

(1) *$a = \bigvee\{b \in L \mid b \ll a\}$ iff $\varphi(a)$ is packed.*
(2) *L is a continuous frame iff X_L is a CL-space.*

Proof (1) By Lemmas 3.6 and 5.2(5),

$$a = \bigvee\{b \in L \mid b \ll a\} \iff \varphi(a) = \mathsf{cl}\ker(a) \iff \ker(a) \text{ is dense in } \varphi(a).$$

(2) This follows from (1). □

It is a well-known fact (see, e.g., [30, p. 289]) that the way below relation on a continuous frame L is interpolating (meaning that $a \ll b$ implies $a \ll c \ll b$ for some $c \in L$). In [37, Lem. 5.3] an alternate proof of this result is given in the language of Priestley spaces:

Lemma 5.6 *Let X be a CL-space and $U, V \in \mathsf{ClopUp}(X)$. If $U \ll V$, then there is $W \in \mathsf{ClopUp}(X)$ such that $U \ll W \ll V$.*

The next lemma is established in [37, Sec. 5] (using different terminology).

Lemma 5.7 *Let X be an L-space and $U, V \in \mathsf{ClopUp}(X)$.*

(1) *If there is a Scott upset F with $U \subseteq F \subseteq V$, then $U \ll V$.*
(2) *If X is a CL-space, then the converse of* (1) *also holds.*

It is well known (see, e.g., [30, p. 311]) that a continuous frame is spatial. In [37, Prop. 4.6] an alternate proof of this result is given in the language of Priestley spaces:

Theorem 5.8 *If X is a CL-space, then X is L-spatial.*

Consequently, if X is a CL-space, then Y is dense in X. We next prove that an L-spatial X is a CL-space iff Y is locally compact. For this we require the following lemma related to the Hofmann–Mislove Theorem [27] (see also [22, Thm. II−1.20]). Recall that the Hofmann–Mislove Theorem establishes a (dual) isomorphism between the poset of compact saturated sets of a sober space Y and the poset of Scott-open filters of the frame of opens of Y. Since Scott upsets correspond to Scott-open filters (see [12, Lem 5.1]), the lemma is in fact a version of the Hofmann–Mislove Theorem in the language of Priestley spaces.

Lemma 5.9 ([12, Thm. 5.7]) *Let L be a frame, X_L its Priestley space, and $Y_L \subseteq X_L$ the localic part of X_L. The map $F \mapsto F \cap Y_L$ is an isomorphism from the poset of Scott upsets of X_L to the poset of compact saturated sets of Y_L (both ordered by inclusion). The inverse isomorphism is given by $K \mapsto \uparrow K$.*

Theorem 5.10 *Let L be a spatial frame, X_L its Priestley space, and $Y_L \subseteq X_L$ the localic part of X_L. Then X_L is a CL-space iff Y_L is locally compact.*

Proof First suppose that X_L is a CL-space, $y \in Y_L$, and $\zeta(a)$ is an open neighborhood of y. Since $\zeta(a) = \varphi(a) \cap Y_L$ (see Lemma 4.1), we have

$$y \in \varphi(a) = \mathsf{cl}\,\ker(a) = \mathsf{cl} \bigcup \{\varphi(b) \mid \varphi(b) \ll \varphi(a)\}.$$

Because $\downarrow y$ is open, $y \in \varphi(b)$ for some $\varphi(b) \ll \varphi(a)$. Therefore,

$$y \in \varphi(b) \cap Y_L = \zeta(b).$$

By Lemma 5.7(2), there is a Scott upset F such that $\varphi(b) \subseteq F \subseteq \varphi(a)$. Thus, $y \in \zeta(b) \subseteq F \cap Y_L \subseteq \zeta(a)$. By Lemma 5.9, $F \cap Y_L$ is compact. Consequently, Y_L is locally compact.

Conversely, suppose that Y_L is locally compact and $a \in L$. We must show that $\ker(a)$ is dense in $\varphi(a)$. Let $x \in \varphi(a)$ and W be an open neighborhood of x in X_L. By Lemma 3.4(1), there exist clopen upsets U and V of X_L such that $x \in U \cap V^c \subseteq W$. Therefore, $U \cap V^c \cap \varphi(a) \neq \varnothing$. Because L is spatial, Y_L is dense in X_L, so $U \cap V^c \cap \zeta(a) \neq \varnothing$, and hence there is $y \in U \cap V^c \cap \zeta(a)$. Since Y_L is locally compact, there is $b \in L$ and a compact saturated $K \subseteq Y_L$ such that $y \in \zeta(b) \subseteq K \subseteq \zeta(a)$. By Lemma 5.9, $\uparrow K$ is a Scott upset. Thus, $\uparrow K$ is closed, and so $\varphi(b) = \mathsf{cl}\,\zeta(b) \subseteq \uparrow K$ by Remark 4.8 (which is applicable since X_L is L-spatial by Theorem 4.7). Therefore, $\varphi(b) \subseteq \uparrow K \subseteq \varphi(a)$. Then $\varphi(b) \ll \varphi(a)$ by Lemma 5.7(1). Thus, $y \in \ker(a)$ by Lemma 5.2(3). This implies that $U \cap V^c \cap \ker(a) \neq \varnothing$, so $\ker(a)$ is dense in $\varphi(a)$. □

Theorems 5.5(2) and 5.10 establish a one-to-one correspondence between continuous frames, CL-spaces, and locally compact sober spaces. Next, we extend these to categorical equivalences.

Lemma 5.11 *Let $h : L_1 \to L_2$ be a frame homomorphism and $f : X_{L_2} \to X_{L_1}$ its dual L-morphism. Then h is proper iff*

$$f^{-1}(\ker U) \subseteq \ker f^{-1}(U) \tag{$\sharp$}$$

for all $U \in \mathsf{ClopUp}(X_{L_1})$.

Proof First suppose that h is proper and $U \in \mathsf{ClopUp}(X_{L_1})$. Let $x \in f^{-1}(\ker U)$. Then $f(x) \in \ker U$. Therefore, there is $V \in \mathsf{ClopUp}(X_{L_1})$ with $f(x) \in V \ll U$. Since $U, V \in \mathsf{ClopUp}(X_{L_1})$, there exist $a, b \in L_1$ with $\varphi(a) = V$ and $\varphi(b) = U$. Then $a \ll b$ by Lemma 5.2(5). Since h is proper, $ha \ll hb$, and hence using Lemma 5.2(5) again, $f^{-1}(V) = \varphi(ha) \ll \varphi(hb) = f^{-1}(U)$. Thus, $x \in f^{-1}(V) \ll f^{-1}(U)$, and so $x \in \ker f^{-1}(U)$.

Conversely, suppose that ($\sharp$) holds for all $U \in \mathsf{ClopUp}(X_{L_1})$. Let $a \ll b$. Then $\varphi(a) \subseteq \ker(b)$ by Lemma 5.2(5). Therefore, $f^{-1}(\varphi(a)) \subseteq f^{-1}(\ker(b))$. Thus, $f^{-1}(\varphi(a)) \subseteq \ker f^{-1}(\varphi(b))$ by ($\sharp$). Consequently, $f^{-1}(\varphi(a)) \ll f^{-1}(\varphi(b))$ by Lemma 5.2(3). Hence, $\varphi(ha) \ll \varphi(hb)$, and so $ha \ll hb$ by Lemma 5.2(5), yielding that h is proper. □

Definition 5.12 Let $f : X_1 \to X_2$ be an L-morphism between L-spaces. We call f *proper* if f satisfies ($\sharp$) for all clopen upsets of X_2.

It is straightforward to check that CL-spaces and proper L-morphisms form a category, which we denote by ConLPries.

Theorem 5.13 ConFrm *is dually equivalent to* ConLPries.

Proof The units $\varphi : L \to \mathscr{D}\mathscr{X}(L)$ and $\varepsilon : X \to \mathscr{X}\mathscr{D}(X)$ of Pultr–Sichler duality (see Remark 3.10) remain isomorphisms in ConFrm and ConLPries. Thus, it follows from Theorem 5.5(2) and Lemma 5.11 that the restrictions of the functors $\mathscr{X}$ and $\mathscr{D}$ yield a dual equivalence between ConFrm and ConLPries. □

We next give several equivalent conditions for an L-morphism between CL-spaces to be proper. For this we require the following lemma, item (1) of which generalizes [37, Lem. 4.5] and provides means to find Scott upsets.

Lemma 5.14 *Let X be a CL-space.*

(1) *Suppose that $\mathcal{U}$ is a down-directed family of clopen upsets of X such that $\bigcap \mathcal{U} = \bigcap\{\ker U \mid U \in \mathcal{U}\}$. Then $\bigcap \mathcal{U}$ is a Scott upset.*
(2) $\ker U = {\uparrow}(U \cap Y)$ *for every* $U \in \mathsf{ClopUp}(X)$.

Proof (1) Clearly $\bigcap \mathcal{U}$ is a closed upset. To see that it is a Scott upset, by Lemma 4.5 it is enough to show that $\bigcap \mathcal{U} \subseteq \mathsf{cl}\, V$ implies $\bigcap \mathcal{U} \subseteq V$ for every open upset V of X. Note that $\mathsf{cl}\, V$ is open since X is an L-space. Therefore, since X is compact and $\bigcap \mathcal{U}$ is down-directed, from $\bigcap \mathcal{U} \subseteq \mathsf{cl}\, V$ it follows that there is $U \in \mathcal{U}$ with $U \subseteq \mathsf{cl}\, V$. Thus, $\ker U \subseteq V$ by Lemma 5.2(4). Since $U \in \mathcal{U}$ and $\bigcap \mathcal{U} = \bigcap\{\ker U \mid U \in \mathcal{U}\}$, we have $\bigcap \mathcal{U} \subseteq \ker U$. Consequently, $\bigcap \mathcal{U} \subseteq V$.

(2) First suppose that $x \in \ker U$. Then there is $V \in \mathsf{ClopUp}(X)$ with $x \in V \ll U$. By Lemma 5.7(2), there is a Scott upset F with $V \subseteq F \subseteq U$. Therefore, there is $y \in F \cap Y$ with $y \le x$. Thus, $x \in {\uparrow}(U \cap Y)$.

Conversely, suppose that $x \in {\uparrow}(U \cap Y)$. Then there is $y \in U \cap Y$ with $y \le x$. Since U is packed, $U = \mathsf{cl} \ker U$, so ${\uparrow} y \subseteq \mathsf{cl} \ker U$. Thus, since ${\uparrow} y$ is a Scott upset and $\ker U$ is an open upset, $x \in {\uparrow} y \subseteq \ker U$ by Lemma 4.5. □

Theorem 5.15 *Let X_1 and X_2 be CL-spaces, Y_1 and Y_2 the localic parts of X_1 and X_2 respectively, and $f : X_1 \to X_2$ an L-morphism. The following are equivalent.*

(1) *f is proper.*
(2) $f^{-1}{\uparrow}(U \cap Y_2) = {\uparrow}(f^{-1}(U) \cap Y_1)$ *for all* $U \in \mathsf{ClopUp}(X_2)$.
(3) $f^{-1}({\uparrow} y)$ *is a Scott upset of* X_1 *for all* $y \in Y_2$.
(4) $f^{-1}(F)$ *is a Scott upset of* X_1 *for all Scott upsets* F *of* X_2.
(5) ${\downarrow} f(x) \cap Y_2 \subseteq {\downarrow} f({\downarrow} x \cap Y_1)$ *for all* $x \in X_1$.

Proof (1)⇒(2) Suppose $x \in f^{-1}{\uparrow}(U \cap Y_2)$. Then $x \in f^{-1}(\ker U)$ by Lemma 5.14(2). Since f is proper, $x \in \ker f^{-1}(U)$, and using Lemma 5.14(2) again yields $x \in {\uparrow}(f^{-1}(U) \cap Y_1)$. For the converse, suppose $x \in {\uparrow}(f^{-1}(U) \cap Y_1)$. Then $x \ge y$ for some $y \in f^{-1}(U) \cap Y_1$. Therefore, $f(x) \ge f(y)$ and $f(y) \in U$. By Lemma 4.12(1), $f(Y_1) \subseteq Y_2$. Thus, $f(y) \in U \cap Y_2$, so $f(x) \in {\uparrow}(Y \cap Y_2)$, and hence $x \in f^{-1}{\uparrow}(Y \cap Y_2)$.

(2)⇒(3) Since ${\uparrow} y$ is a closed upset, ${\uparrow} y = \bigcap\{U \in \mathsf{ClopUp}(X_2) \mid y \in U\}$ by Lemma 3.4(3). Therefore, we have ${\uparrow} y = \bigcap\{{\uparrow}(U \cap Y_2) \mid y \in U \in \mathsf{ClopUp}(X_2)\}$ since $y \in Y_2$. Thus, by (2) and Lemma 5.14(2),

$$\bigcap\{f^{-1}(U) \mid y \in U \in \mathsf{ClopUp}(X_2)\} = f^{-1}\left(\bigcap\{U \in \mathsf{ClopUp}(X_2) \mid y \in U\}\right)$$

$$= f^{-1}\left(\bigcap\{\uparrow(U \cap Y_2) \mid y \in U \in \mathsf{ClopUp}(X_2)\}\right)$$
$$= \bigcap\{f^{-1}\uparrow(U \cap Y_2) \mid y \in U \in \mathsf{ClopUp}(X_2)\}$$
$$= \bigcap\{\uparrow(f^{-1}(U) \cap Y_1) \mid y \in U \in \mathsf{ClopUp}(X_2)\}$$
$$= \bigcap\{\ker f^{-1}(U) \mid y \in U \in \mathsf{ClopUp}(X_2)\}.$$

Consequently,

$$f^{-1}(\uparrow y) = \bigcap\{f^{-1}(U) \mid y \in U \in \mathsf{ClopUp}(X_2)\}$$
$$= \bigcap\{\ker f^{-1}(U) \mid y \in U \in \mathsf{ClopUp}(X_2)\}$$

is a Scott upset by Lemma 5.14(1).

(3)⇒(4) Let F be a Scott upset of X_2. By (3) we have

$$\min f^{-1}(F) = \min f^{-1}\bigcup\{\uparrow y \mid y \in \min F\}$$
$$= \min\bigcup\{f^{-1}(\uparrow y) \mid y \in \min F\}$$
$$\subseteq \bigcup\{\min f^{-1}(\uparrow y) \mid y \in \min F\} \subseteq Y_1.$$

Thus, $f^{-1}(F)$ is a Scott upset of X_1.

(4)⇒(5) Suppose $y_2 \in \downarrow f(x) \cap Y_2$. Then $\uparrow y_2$ is a Scott upset of X_2, so $f^{-1}(\uparrow y_2)$ is a Scott upset of X_1 by (4). Since $x \in f^{-1}(\uparrow y_2)$, there is $y_1 \in \min f^{-1}(\uparrow y_2)$ such that $y_1 \le x$. Therefore, $y_2 \le f(y_1)$ and $y_1 \in \downarrow x \cap Y_1$. Thus, $y_2 \in \downarrow f(\downarrow x \cap Y_1)$.

(5)⇒(1) Let $x \in f^{-1}(\ker U)$. Then $f(x) \in \ker(U)$, and Lemma 5.14(2) implies that $f(x) \in \uparrow(U \cap Y_2)$. Therefore, there is $y \in \downarrow f(x) \cap (U \cap Y_2)$. By (5), $y \in \downarrow f(\downarrow x \cap Y_1)$, so there is $y' \in \downarrow x \cap Y_1$ with $y \le f(y')$. Thus, $f(y') \in U$, and hence $y' \in f^{-1}(U) \cap Y_1$. Consequently, Lemma 5.14(2) yields that $x \in \uparrow(f^{-1}(U) \cap Y_1) = \ker(f^{-1}(U) \cap Y_1) \subseteq \ker f^{-1}(U)$. □

Let $h : L_1 \to L_2$ be a frame homomorphism between continuous frames, $f : X_{L_2} \to X_{L_1}$ its dual L-morphism, and $\mathscr{Y} f : Y_{L_2} \to Y_{L_1}$ the restriction of f. The next theorem characterizes when each of these maps is proper.

Theorem 5.16 *The following are equivalent.*

(1) *$h : L_1 \to L_2$ is a proper frame homomorphism.*
(2) *$f : X_{L_2} \to X_{L_1}$ is a proper L-morphism.*
(3) *$\mathscr{Y} f : Y_{L_2} \to Y_{L_1}$ is a proper map.*

Proof (1)⇔(2) This follows from Lemma 5.11.

(2)⇒(3) We let $g = \mathscr{Y} f$ and verify that g satisfies Definition 2.4(2). By Remark 2.5(1), it is sufficient to show that $g^{-1}(U)$ is compact for each compact saturated U in Y_{L_1}. Since U is compact saturated in Y_{L_1}, we have that $\uparrow U$ is a Scott upset of X_{L_1} by Lemma 5.9. Hence, $f^{-1}(\uparrow U)$ is a Scott upset of X_{L_2} by Theorem 5.15(4). Thus, $f^{-1}(\uparrow U) \cap Y_{L_2}$ is compact saturated in Y_{L_2} by Lemma 5.9. But $f^{-1}(\uparrow U) \cap Y_{L_2} = g^{-1}(U)$ because U is saturated in Y_{L_2} and g is the restriction of f to Y_{L_2}. Therefore, $g^{-1}(U)$ is compact.

(3)⇒(2) By Theorem 5.15(3), it is enough to show that $f^{-1}(\uparrow y)$ is a Scott upset of X_{L_2} for each $y \in Y_{L_1}$. Since $y \in Y_{L_1}$, we have that $\uparrow y$ is a Scott upset of X_{L_1}, so $\uparrow y \cap Y_{L_1}$ is compact saturated in Y_{L_1} by Lemma 5.9. Let $g = \mathscr{Y} f$. Because g is proper, $g^{-1}(\uparrow y \cap Y_{L_1})$ is compact saturated in Y_{L_2}. Hence, $\uparrow g^{-1}(\uparrow y \cap Y_{L_1})$ is a Scott upset of X_{L_2} by Lemma 5.9. Therefore, it suffices to show that $f^{-1}(\uparrow y) = \uparrow g^{-1}(\uparrow y \cap Y_{L_1})$.

Clearly $\uparrow g^{-1}(\uparrow y \cap Y_{L_1}) \subseteq f^{-1}(\uparrow y)$. For the reverse inclusion, let $x \notin \uparrow g^{-1}(\uparrow y \cap Y_{L_1})$. Then there is a clopen downset D of X_{L_2} such that $x \in D$ and $D \cap g^{-1}(\uparrow y \cap Y_{L_1}) = \varnothing$. Hence, $y \notin \downarrow g(D \cap Y_{L_2})$. Since g is proper, $\downarrow g(D \cap Y_{L_2}) \cap Y_{L_1}$ is closed in Y_{L_1}, and so $\downarrow g(D \cap Y_{L_2}) \cap Y_{L_1} = E \cap Y_{L_1}$ for some clopen downset E of X_{L_1}. Therefore, $y \notin E$ and $g(D \cap Y_{L_2}) \subseteq E$, so $\downarrow \mathsf{cl}\, g(D \cap Y_{L_2}) \subseteq E$. Because X_{L_1} and X_{L_2} are CL-spaces, they are L-spatial by Theorem 5.8. Thus, since f is a closed map, we have

$$\downarrow f(D) = \downarrow f\, \mathsf{cl}(D \cap Y_{L_2}) = \downarrow \mathsf{cl}\, f(D \cap Y_{L_2}) = \downarrow \mathsf{cl}\, g(D \cap Y_{L_2}) \subseteq E.$$

Consequently, $y \notin \downarrow f(D)$, and hence $x \notin f^{-1}(\uparrow y)$. □

Corollary 5.17 *Suppose X_1, X_2 are CL-spaces and $g : Y_1 \to Y_2$ is a proper map between their localic parts. Then there is a proper L-morphism $f : X_1 \to X_2$ extending g.*

Proof By Lemma 4.17, there is an L-morphism $f : X_1 \to X_2$ extending g. Thus, $\mathscr{Y} f = g$, and so f is proper by Theorem 5.16. □

Theorem 5.18 $\mathsf{ConLPries}$ *is equivalent to* LKSob.

Proof It follows from Theorem 5.10 that the restriction of $\mathscr{Y}$ is well defined on objects. By Theorem 5.16, the restriction of $\mathscr{Y}$ is also well defined on morphisms. This together with Corollary 5.17 shows that Theorems 4.13 and 4.18 apply to yield that the restriction $\mathscr{Y} : \mathsf{ConLPries} \to \mathsf{LKSob}$ is essentially surjective, full, and faithful. □

We thus obtain the following alternative proof of Hofmann–Lawson duality:

Corollary 5.19 (Hofmann–Lawson) ConFrm *is dually equivalent to* LKSob.

Proof Combine Theorems 5.13 and 5.18. □

6 Deriving Dualities for Stably Continuous Frames

To derive the two dualities for stably continuous frames, we first characterize stability of $\ll$ in the language of Priestley spaces.

Lemma 6.1 *Let L be a continuous frame, X_L its Priestley space, and $Y_L \subseteq X_L$ the localic part of X_L. For $a, b \in L$ we have*

$$(\forall c \in L)(c \ll a, b \Rightarrow c \ll a \wedge b) \text{ iff } \ker(a) \cap \ker(b) = \ker(a \wedge b).$$

Proof First suppose that $(\forall c \in L)(c \ll a, b \Rightarrow c \ll a \wedge b)$. Then Lemma 5.2(5) gives

$$(\forall c \in L)(\varphi(c) \ll \varphi(a), \varphi(b) \implies \varphi(c) \ll \varphi(a) \cap \varphi(b)). \qquad (\star)$$

Since ker is monotone (see Lemma 5.2(2)), $\ker(a \wedge b) \subseteq \ker(a) \cap \ker(b)$. For the reverse inclusion, let $x \in \ker(a) \cap \ker(b)$. Then there are $d, e \in L$ with $x \in \varphi(d) \ll \varphi(a)$ and $x \in \varphi(e) \ll \varphi(b)$. Let $c = d \wedge e$. Then $x \in \varphi(c) \ll \varphi(a), \varphi(b)$. Consequently, by ($\star$), $\varphi(c) \ll \varphi(a) \cap \varphi(b) = \varphi(a \wedge b)$. Therefore, $x \in \ker(a \wedge b)$ by Lemma 5.2(5). Thus, $\ker(a) \cap \ker(b) = \ker(a \wedge b)$.

For the converse, suppose that $\ker(a) \cap \ker(b) = \ker(a \wedge b)$. Let $c \in L$ with $c \ll a, b$. Then $\varphi(c) \subseteq \ker(a), \ker(b)$ by Lemma 5.2(5). Therefore, $\varphi(c) \subseteq \ker(a) \cap \ker(b) = \ker(a \wedge b)$. Thus, using Lemma 5.2(5) again, we obtain $c \ll a \wedge b$. □

The previous lemma motivates defining stability in terms of kernels commuting with intersections. We will see in Lemma 6.3 that for CL-spaces this property coincides with the property that binary intersections of Scott upsets are Scott upsets.

Definition 6.2 Let X be an L-space.

(1) We call X *kernel-stable* if for all clopen upsets U and V of X we have

$$\ker U \cap \ker V = \ker(U \cap V).$$

(2) We call X *Scott-stable* if for all Scott upsets F and G of X we have that $F \cap G$ is a Scott upset.

Lemma 6.3 *Let X be a CL-space.*

(1) *For every Scott upset F, we have $F = \bigcap\{\ker U \mid F \subseteq U \in \mathsf{ClopUp}(X)\}$.*
(2) *X is kernel-stable iff X is Scott-stable.*

Proof (1) Suppose $F \subseteq U \in \mathsf{ClopUp}(X)$. Since X is a CL-space, $F \subseteq \mathsf{cl}\ker U$. By Lemma 5.2(1), $\ker U$ is an open upset. Therefore, $F \subseteq \ker U$ by Lemma 4.5. Thus, we have that $F \subseteq \bigcap\{\ker U \mid F \subseteq U \in \mathsf{ClopUp}(X)\}$. For the reverse inclusion, by Lemma 3.4(3), we have

$$F = \bigcap\{U \mid F \subseteq U \in \mathsf{ClopUp}(X)\} \supseteq \bigcap\{\ker U \mid F \subseteq U \in \mathsf{ClopUp}(X)\}.$$

(2) Suppose X is kernel-stable. Let F, G be Scott upsets. If U, V, W range over clopen upsets of X, by (1) we have

$$\begin{aligned} F \cap G &= \bigcap\{\ker U \mid F \subseteq U\} \cap \bigcap\{\ker V \mid G \subseteq V\} \\ &= \bigcap\{\ker U \cap \ker V \mid F \subseteq U, G \subseteq V\} \\ &= \bigcap\{\ker(U \cap V) \mid F \subseteq U, G \subseteq V\} \\ &= \bigcap\{\ker W \mid F \cap G \subseteq W\} \\ &\subseteq \bigcap\{W \mid F \cap G \subseteq W\} = F \cap G, \end{aligned}$$

where the last equality follows from Lemma 3.4(3); for the second to last equality it is enough to observe that by compactness, $F \cap G \subseteq W$ is equivalent to $U \cap V \subseteq W$ for some clopen upsets $U \supseteq F$ and $V \supseteq G$. Thus, $F \cap G$ is a Scott upset by Lemma 5.14(1), and hence X is Scott-stable.

Conversely, suppose X is Scott-stable. Let U, V be clopen upsets. Since $\ker U$ is an open upset for each U (see Lemma 5.2(1)), it suffices to show that $W \subseteq \ker(U) \cap \ker(V)$ iff $W \subseteq \ker(U \cap V)$ for each clopen upset W (see Lemma 3.4(2)). Let W be a clopen upset. By Lemma 5.2(3), $W \subseteq \ker(U) \cap \ker(V)$ iff $W \ll U, V$. By Lemma 5.7, this happens iff there are Scott upsets F and G such that $W \subseteq F \subseteq U$ and $W \subseteq G \subseteq V$. Since X is Scott-stable, the latter is equivalent to the existence of a Scott upset H such that $W \subseteq H \subseteq U \cap V$. By invoking Lemma 5.7 again, this is equivalent to $W \ll U \cap V$, which in turn is equivalent to $W \subseteq \ker(U \cap V)$ by Lemma 5.2(3). Thus, X is kernel-stable. □

Theorem 6.4 *Let L be a frame and X_L its Priestley space. Then L is a stably continuous frame iff X_L is a Scott-stable CL-space.*

Proof Apply Theorem 5.5(2), Lemma 6.1, and Lemma 6.3(2). □

Definition 6.5 We call an L-space *stably continuous* or simply a *StCL-space* if it is a Scott-stable CL-space. Let $\mathsf{StCLPries}$ be the full subcategory of $\mathsf{ConLPries}$ consisting of StCL-spaces.

Corollary 6.6 StCFrm *is dually equivalent to* $\mathsf{StCLPries}$.

Proof Restrict Theorem 5.13 to the full subcategories StCFrm and $\mathsf{StCLPries}$ using Theorem 6.4. □

Next we show that $\mathsf{StCLPries}$ is equivalent to StLKSp.

Theorem 6.7 *Let X be an SL-space and Y its localic part. Then X is a StCL-space iff Y is stably locally compact.*

Proof Suppose X is a StCL-space. By Theorem 5.10, Y is locally compact. Let K, J be compact saturated sets in Y. Then $\uparrow K$ and $\uparrow J$ are Scott upsets by Lemma 5.9. Since X is Scott-stable, $\uparrow K \cap \uparrow J$ is a Scott upset. Therefore, $K \cap J = \uparrow K \cap \uparrow J \cap Y$ is compact saturated by Lemma 5.9.

Conversely, suppose that Y is stably locally compact. By Theorem 5.10, X is a CL-space. By Lemma 6.3(2), it is enough to show that X is kernel-stable. Let $U, V \in \mathsf{ClopUp}(X)$. Since ker is monotone by Lemma 5.2(2), it suffices to show that $\ker U \cap \ker V \subseteq \ker(U \cap V)$. Let $x \in \ker U \cap \ker V$. Then there exist $U', V' \in \mathsf{ClopUp}(X)$ containing x such that $U' \ll U$ and $V' \ll V$. By Lemma 5.7(2), there are Scott upsets F, G with $U' \subseteq F \subseteq U$ and $V' \subseteq G \subseteq V$. By Lemma 5.9, $F \cap Y$ and $G \cap Y$ are compact saturated. Since Y is stably locally compact, $F \cap G \cap Y$ is compact saturated. Hence, $\uparrow(F \cap G \cap Y)$ is a Scott upset by Lemma 5.9. Moreover, because $F \cap G \subseteq U \cap V$, we have $\uparrow(F \cap G \cap Y) \subseteq \ker(U \cap V)$ by Lemma 5.14(2). Therefore, since X is L-spatial, $x \in U' \cap V' = \mathsf{cl}(U' \cap V' \cap Y) \subseteq \uparrow(F \cap G \cap Y) \subseteq \ker(U \cap V)$. □

Corollary 6.8 $\mathsf{StCLPries}$ *is equivalent to* StLKSp.

Proof Use Theorem 6.7 to restrict Theorem 5.18 to the full subcategories $\mathsf{StCLPries}$ and StLKSp. □

As a consequence of Corollaries 6.6 and 6.8, we obtain the following well-known duality for stably continuous frames (see Theorem 2.6(3)):

Corollary 6.9 StCFrm *is dually equivalent to* StLKSp.

We next turn our attention to stably compact frames.

Lemma 6.10 *Let L be a frame and X_L its Priestley space. For $a \in L$, the following are equivalent.*

(1) *a is compact.*
(2) $\ker(a) = \varphi(a)$.
(3) *$\varphi(a)$ is a Scott upset.*

In particular, L is compact iff $X_L = \ker X_L$ iff X_L is a Scott upset.

Proof (1)⇒(2) This follows from Lemma 5.2(1) and (5).

(2)⇒(3) Suppose $\varphi(a) \subseteq \mathsf{cl}\, U$ for some open upset U of X_L. Then $\varphi(a) = \ker(a) \subseteq U$ by Lemma 5.2(4). Therefore, $\varphi(a)$ is a Scott upset by Lemma 4.5.

(3)⇒(1) Since $\varphi(a)$ is a Scott upset, $\varphi(a) \ll \varphi(a)$ by Lemma 5.7(1). Thus, $a \ll a$ by Lemma 5.2(5).

The last statement follows from the above equivalence and the fact that $X_L = \varphi(1)$. □

Remark 6.11 The equivalence (1)⇔(3) of Lemma 6.10 is known (see, e.g., [12, Cor. 5.4]), and so is the fact that L is compact iff X_L is a Scott upset (see [36, Thm. 3.5] or [9, Lem. 3.1]). To this Lemma 6.10 adds a characterization in terms of kernels.

Definition 6.12 Let X be an L-space.

(1) We call X *L-compact* if X is a Scott upset.
(2) We call X *stably L-compact* or simply a *StKL-space* if X is an L-compact StCL-space.
(3) Let $\mathsf{StKLPries}$ be the full subcategory of $\mathsf{StCLPries}$ consisting of StKL-spaces.

As an immediate consequence of Theorem 6.4 and Lemma 6.10, we obtain:

Theorem 6.13 *Let L be a frame and X_L its Priestley space. Then L is a stably compact frame iff X_L is a StKL-space.*

This together with Corollary 6.6 yields:

Corollary 6.14 StKFrm *is dually equivalent to* $\mathsf{StKLPries}$.

To connect StKL-spaces with stably compact spaces, we need the following lemma.

Lemma 6.15 *Let X be an L-space and Y the localic part of X. If X is L-compact, then Y is compact. If in addition X is L-spatial, then the converse also holds.*

Proof Suppose X is L-compact. Let $\mathcal{U}$ be an open cover of Y. For each $U' \in \mathcal{U}$ there is a clopen upset U of X such that $U \cap Y = U'$. Since X is L-compact, $\min(X) \subseteq Y \subseteq \bigcup\{U \mid U' \in \mathcal{U}\}$. Therefore, $X = {\uparrow}(\min X) \subseteq \bigcup\{U \mid U' \in \mathcal{U}\}$ because the latter is an upset. Since X is compact, there are $U_1', \dots, U_n' \in \mathcal{U}$ such that $X \subseteq U_1 \cup \dots \cup U_n$. But then $Y \subseteq (U_1 \cup \dots \cup U_n) \cap Y = U_1' \cup \dots \cup U_n'$. Thus, Y is compact.

Conversely, suppose X is not L-compact, so there is $x \in \min(X) \setminus Y$. Then ${\downarrow}x = \{x\}$ is not open. Hence, $U = \{x\}^c$ is an open upset such that $x \in \mathsf{cl}\, U$. Since U is an open upset, $U = \bigcup\{V \in \mathsf{ClopUp}(X) \mid x \notin V\}$ by Lemma 3.4(2). Therefore,

$$X = \mathsf{cl}\, U = \mathsf{cl} \bigcup \{V \in \mathsf{ClopUp}(X) \mid x \notin V\},$$

and hence $Y \subseteq \bigcup\{V \in \mathsf{ClopUp}(X) \mid x \notin V\}$ by Lemma 4.14(1). If Y were compact, there would exist a clopen upset V such that $x \notin V$ and $Y \subseteq V$. Since X is L-spatial, this would imply $x \in X = \mathsf{cl}\, Y \subseteq V$, a contradiction. Thus, Y is not compact. □

Theorem 6.16 *Let X be an SL-space and Y its localic part. Then X is a StKL-space iff Y is stably compact.*

Proof Apply Theorem 6.7 and Lemma 6.15. □

Corollary 6.17 $\mathsf{StKLPries}$ *is equivalent to* StKSp.

Proof Apply Corollary 6.8 and Theorem 6.16. □

As a consequence of Colloraries 6.14 and 6.17, we obtain the following well-known duality for stably compact frames (see Theorem 2.6(4)):

Corollary 6.18 StKFrm *is dually equivalent to* StKSp.

7 Deriving Isbell Duality

A characterization of the well inside relation $\prec$ on a frame L in the language of the Priestley space of L was given in [9, Sec. 3]. Similar to the notion of the kernel of a clopen upset U, which we introduced in Sect. 3, the well inside relation induces the notion of the regular part of U.

Definition 7.1 Let X be an L-space. For $U, V \in \mathsf{ClopUp}(X)$ we write $U \prec V$ provided ${\downarrow}U \subseteq V$. Define the *regular part* of U as

$$\operatorname{reg} U = \bigcup\{V \in \mathsf{ClopUp} \mid V \prec U\}.$$

When $U = \varphi(a)$, we write $\operatorname{reg}(a)$ for $\operatorname{reg} U$.

This definition is motivated by the following:

Lemma 7.2 ([9, Sec. 3]) *Let X be an L-space. For each $U \in \mathsf{ClopUp}(X)$ we have*

$$\operatorname{reg} U = X \setminus {\downarrow}{\uparrow}(X \setminus U).$$

In particular, if X is the Priestley space of a frame L, then for $a, b \in L$ we have:

$$a \prec b \text{ iff } \varphi(a) \prec \varphi(b) \text{ iff } \varphi(a) \subseteq \operatorname{reg}(b).$$

Consequently, a frame L is regular iff in its Priestley space the regular part of each clopen upset U is dense in U (see [9, Lem. 3.6]). We will give several equivalent characterizations for this condition in Lemma 7.4. For this we need the following:

Lemma 7.3 *Let X be an L-space, $x \in X$, $Z \subseteq X$, and $U \in \mathsf{ClopUp}(X)$.*

(1) $x \in \operatorname{reg} U$ *iff* ${\downarrow}{\uparrow}x \subseteq U$.
(2) $Z \subseteq \operatorname{reg} U$ *iff* ${\downarrow}{\uparrow}Z \subseteq U$

Proof (1) By Lemma 7.2,

$$\begin{aligned} x \in \operatorname{reg} U &\iff x \notin {\downarrow}{\uparrow}(X \setminus U) \iff {\uparrow}x \cap {\uparrow}(X \setminus U) = \varnothing \\ &\iff {\downarrow}{\uparrow}x \cap U^c = \varnothing \iff {\downarrow}{\uparrow}x \subseteq U. \end{aligned}$$

(2) This follows from (1) since ${\downarrow}{\uparrow}Z = \bigcup\{{\downarrow}{\uparrow}x \mid x \in Z\}$. □

Lemma 7.4 *Let X be an L-space, Y the localic part of X, and $U \in \mathsf{ClopUp}(X)$. The following three conditions are equivalent.*

(1) $U \cap Y \subseteq \operatorname{reg} U$.
(2) *for each $y \in U \cap Y$ there are disjoint clopen upsets V, W such that $y \in V$ and $U^c \subseteq W$.*
(3) ${\downarrow}{\uparrow}(U \cap Y) \subseteq U$.

Moreover, the following condition implies conditions (1)–(3).

(4) $\operatorname{reg} U$ *is dense in* U.

Furthermore, if X is L-spatial, all four conditions are equivalent.

Proof (1)⇒(2) Let $y \in U \cap Y$. By (1), $y \in \operatorname{reg} U$. Hence, there is a clopen upset V such that $y \in V$ and $\downarrow V \subseteq U$. Since X is an Esakia space, $\downarrow V$ is a clopen downset. Therefore, $W := (\downarrow V)^c$ is a clopen upset disjoint from V with $U^c \subseteq W$.

(2)⇒(3) Let $x \in \downarrow\uparrow(U \cap Y)$. Then there is $y \in U \cap Y$ such that $x \in \downarrow\uparrow y$. By (2), there are disjoint clopen upsets V, W such that $y \in V$ and $U^c \subseteq W$. Thus, $x \in \downarrow\uparrow y \subseteq \downarrow V \subseteq W^c \subseteq U$.

(3)⇒(1) Apply Lemma 7.3(2).

Therefore, conditions (1)–(3) are equivalent.

(4)⇒(1) Let $y \in U \cap Y$. Then $y \in \mathsf{cl} \operatorname{reg} U$ by (4). Since $\downarrow y$ is open and $\operatorname{reg} U$ is an upset, we conclude that $y \in \operatorname{reg} U$.

Let X be L-spatial.

(1)⇒(4) From $U \cap Y \subseteq \operatorname{reg} U$ it follows that $\mathsf{cl}(U \cap Y) \subseteq \mathsf{cl} \operatorname{reg} U$. But $\mathsf{cl}(U \cap Y) = U$ since X is L-spatial. Thus, $\operatorname{reg} U$ is dense in U. □

Remark 7.5 Compare Lemma 7.4(2) to the usual definition of regularity in topological spaces.

Definition 7.6 Let X be an L-space.

(1) A clopen upset U of X is *L-regular* if $\operatorname{reg} U$ is dense in U.
(2) X is *L-regular* if all its clopen upsets are L-regular.
(3) X is a *KRL-space* if X is L-compact and L-regular.

Let $\mathsf{KRLPries}$ be the full subcategory of LPries consisting of KRL-spaces. The next theorem goes back to [9, Sec. 3] (see also [36, Sec. 3]).

Theorem 7.7 *Let L be a frame and X_L its Priestley space.*

(1) ([9, Lem. 3.6]) *L is regular iff X_L is L-regular.*
(2) ([9, Thm. 3.9]) *L is compact regular iff X_L is a KRL-space.*

Corollary 7.8 KRFrm *is dually equivalent to* $\mathsf{KRLPries}$.

Proof Apply Theorems 3.9 and 7.7(2). □

Lemma 7.9 *Let X be a KRL-space and Y the localic part of X. Then X is L-spatial and Y is compact.*

Proof By Lemma 6.15, it is sufficient to show that X is L-spatial, for which, by Lemma 3.4(1), it is enough to show that $U \cap V^c \neq \varnothing$ implies $U \cap V^c \cap Y \neq \varnothing$ for all $U, V \in \mathsf{ClopUp}(X)$. Since X is L-regular, $U = \mathsf{cl} \operatorname{reg} U$. Therefore, $U \cap V^c \neq \varnothing$ implies $\operatorname{reg} U \cap V^c \neq \varnothing$. Let $z \in \operatorname{reg} U \cap V^c$. Because $z \in \operatorname{reg} U$, there is a clopen upset W containing z such that $\downarrow W \subseteq U$. By Lemma 3.4(5), there is $y \in \min(\downarrow W)$ with $y \leq z$. Consequently, $y \in \downarrow W \subseteq U$ and $y \in V^c$ since V^c is a downset. Moreover, $y \in Y$ because $y \in \min(X)$ and $\min(X) \subseteq Y$ since X is L-compact. □

Remark 7.10 Lemma 7.9 provides an alternative proof of the well-known fact that each compact regular frame is spatial (see, e.g., [30, p. 90]).

Theorem 7.11 *Let X be an SL-space and Y the localic part of X. Then X is a KRL-space iff Y is compact Hausdorff.*

Proof Let X be a KRL-space. By Lemma 7.9, Y is compact. We prove that Y is regular. Let $y \in Y$ and F be a closed subset of Y with $y \notin F$. Then $Y \backslash F$ is an open subset of Y containing y. Therefore, there is a clopen upset U of X with $U \cap Y = Y \setminus F$. Since X is L-regular, U is L-regular. Therefore, by the implication (4)⇒(2) in Lemma 7.4, there exist disjoint clopen upsets V, W such that $y \in V$ and $U^c \subseteq W$. Thus, $V \cap Y$ and $W \cap Y$ are disjoint open subsets of Y such that $y \in V \cap Y$ and $F = Y \backslash U \subseteq W \cap Y$. This implies that Y is regular. Consequently, Y is compact Hausdorff.

Conversely, let Y be compact Hausdorff. Since X is an SL-space, X is L-compact by Lemma 6.15. To see that X is L-regular, let U be a clopen upset of X. Suppose $y \in U \cap Y$. Since Y is regular, $Y \backslash U$ is closed in Y, and $y \notin Y \backslash U$, there exist clopen upsets V, W such that $V \cap W \cap Y = \varnothing$, $y \in V$, and $Y \backslash U \subseteq W \cap Y$. Therefore, since X is L-spatial,

$$V \cap W = \mathsf{cl}(V \cap Y) \cap \mathsf{cl}(W \cap Y) = \mathsf{cl}(V \cap W \cap Y) = \mathsf{cl}\,\varnothing = \varnothing,$$

where the second equality follows from Lemma 4.15. Moreover, $Y \backslash U \subseteq W \cap Y$ implies $U^c \subseteq W$ because X is L-spatial. Thus, U is L-regular by Lemma 7.4. This finishes the proof that X is a KRL-space. □

Corollary 7.12 KRLPries *is equivalent to* KHaus.

Proof Apply Corollary 4.19, Lemma 7.9, and Theorem 7.11. □

We can now derive Isbell duality from Corollaries 7.8 and 7.12:

Corollary 7.13 KRFrm *is dually equivalent to* KHaus.

We next compare ker with reg. This is reminiscent of the comparison between compact and complemented elements in frames.

Lemma 7.14 *Let X be an L-space.*

(1) *If $U \in \mathsf{ClopUp}(X)$ is L-regular, then* $\ker U \subseteq \operatorname{reg} U$.
(2) *X is L-compact iff* $\operatorname{reg} U \subseteq \ker U$ *for every* $U \in \mathsf{ClopUp}(X)$.
(3) *If X is a KRL-space, then* $\operatorname{reg} U = \ker U$ *for every* $U \in \mathsf{ClopUp}(X)$.

Proof (1) Since U is L-regular, $\mathsf{cl}\operatorname{reg} U = U$. Therefore, $\ker U \subseteq \operatorname{reg} U$ by Lemma 5.2(4).

(2) First suppose that X is L-compact and $U \in \mathsf{ClopUp}(X)$. We show that $V \prec U$ implies $V \ll U$ for every $V \in \mathsf{ClopUp}(X)$. Let $U \subseteq \mathsf{cl}(W)$ for some open upset W. Then $U \cap Y \subseteq W$ by Lemma 4.14(1). Moreover, since ${\downarrow}V \subseteq U$, we have $\min({\downarrow}V) \subseteq U$. Therefore, $\min({\downarrow}V) \subseteq U \cap Y$ because X is L-compact. Thus, $V \subseteq {\uparrow}\min({\downarrow}V) \subseteq {\uparrow}(U \cap Y) \subseteq W$. Consequently, $V \ll U$, and hence $\operatorname{reg} U \subseteq \ker U$.

Conversely, since $\operatorname{reg} X = X$ (see Lemma 7.2), $\operatorname{reg} X \subseteq \ker X$ implies $\ker X = X$, so X is L-compact by Lemma 6.10.

(3) This follows from (1) and (2). □

For the next lemma we recall from Sect. 2 that a subset of a poset is a biset if it is both an upset and a downset.

Lemma 7.15 *Let X be an L-space and Y the localic part of X.*

(1) *If X is L-compact, then each closed biset is a Scott upset.*
(2) *If X is L-regular, then each Scott upset is a biset.*
(3) *If X is L-regular, then* $Y \subseteq \min(X)$.

(4) *If X is a KRL-space, then closed bisets are exactly Scott upsets.*
(5) *If X is a KRL-space, then* $\min(X) = Y$. *Consequently,* $\min(\downarrow F) = \downarrow F \cap Y$ *for every* $F \subseteq X$.

Proof (1) Since X is L-compact, $\min(X) \subseteq Y$. Therefore, for each closed biset F, we have $\min(F) \subseteq \min(X) \subseteq Y$. Thus, F is a Scott upset.

(2) Suppose F is a Scott upset. Let $x \in \downarrow F$. Then there is $z \in F$ with $x \leq z$. Since F is a Scott upset, there is $y \in F \cap Y$ with $y \leq z$. If $y \not\leq x$, then there is a clopen upset U with $y \in U$ and $x \notin U$. Since X is L-regular, we have $\downarrow\uparrow(U \cap Y) \subseteq U$ by the implication (4)⇒(3) in Lemma 7.4. Therefore, $\downarrow\uparrow y \subseteq U$, so $x \in U$, a contradiction. Thus, we must have $y \leq x$, so $x \in F$, and hence F is a biset.

(3) Let $y \in Y$. Then $\uparrow y$ is a Scott upset, so $\uparrow y$ is a downset by (2). Thus, $y \in \min(X)$.

(4) This follows from (1) and (2).

(5) Since X is a KRL-space, X is L-compact, so $\min(X) \subseteq Y$. The reverse inclusion follows from (3). Consequently, $\min(\downarrow F) = \downarrow F \cap \min(X) = \downarrow F \cap Y$. □

Remark 7.16 The frame-theoretic reading of the first part of Lemma 7.15(5) is that in a compact regular frame the minimal prime filters are exactly the completely prime filters. This was first observed in [9, Lem. 5.2, 5.3].

Theorem 7.17 *Each KRL-space is a StKL-space.*

Proof Let X be a KRL-space. We first prove that X is a CL-space. Let U be a clopen upset of X. By Lemma 7.14(3), $\operatorname{reg} U = \ker U$, so $\mathsf{cl} \ker U = \mathsf{cl} \operatorname{reg} U = U$ since U is a L-regular. Therefore, X is a CL-space.

Next let U, V be clopen upsets of X. For each clopen upset W, by Lemma 7.14(3) we have

$$\begin{aligned} W \subseteq \ker U \cap \ker V &\iff W \subseteq \operatorname{reg} U \cap \operatorname{reg} V \iff W \subseteq \operatorname{reg} U, \operatorname{reg} V \\ &\iff \downarrow W \subseteq U \cap V \iff W \subseteq \operatorname{reg}(U \cap V) \\ &\iff W \subseteq \ker(U \cap V). \end{aligned}$$

Therefore, $\ker U \cap \ker V = \ker(U \cap V)$, and so X is kernel-stable. Since X is a CL-space, X is Scott-stable by Lemma 6.3(2). Also, because X is a KRL-space, X is L-compact. Consequently, X is a StKL-space. □

Theorem 7.18 *Let $f : X_1 \to X_2$ be an L-morphism between L-spaces.*

(1) $f^{-1}(\operatorname{reg} U) \subseteq \operatorname{reg} f^{-1}(U)$ *for each clopen upset U of X_2.*
(2) *If X_1 is L-compact and X_2 is L-regular, then f is proper.*

Proof (1) Suppose $x \in f^{-1}(\operatorname{reg} U)$. Then $f(x) \in \operatorname{reg} U$. Therefore, $\downarrow\uparrow f(x) \subseteq U$ by Lemma 7.3(1). Since f is order-preserving, we obtain $f(\downarrow\uparrow x) \subseteq U$. Thus, $\downarrow\uparrow x \subseteq f^{-1}(U)$, and so $x \in \operatorname{reg} f^{-1}(U)$ by Lemma 7.3(1).

(2) Let U be a clopen upset of X_2. Since X_2 is L-regular, U is L-regular. Therefore, since X_1 is L-compact, by (1) and Lemma 7.14, we have

$$f^{-1}(\ker U) \subseteq f^{-1}(\operatorname{reg} U) \subseteq \operatorname{reg} f^{-1}(U) \subseteq \ker f^{-1}(U).$$

Thus, f is proper. □

Remark 7.19 Theorem 7.18(2) corresponds to the well-known fact that every frame homomorphism from a compact frame to a regular frame is proper.

Fig. 2 Equivalences and dual equivalences between various categories of frames, L-spaces, and sober spaces

$$
\begin{array}{ccccc}
\mathsf{DLat} & \leftrightsquigarrow & \mathsf{Pries} & & \\
\curlyvee & & \curlyvee & & \\
\mathsf{Frm} & \leftrightsquigarrow & \mathsf{LPries} & & \\
\vee & & \vee & & \\
\mathsf{SFrm} & \leftrightsquigarrow & \mathsf{SLPries} & \longleftrightarrow & \mathsf{Sob} \\
\curlyvee & & \curlyvee & & \curlyvee \\
\mathsf{ConFrm} & \leftrightsquigarrow & \mathsf{ConLPries} & \longleftrightarrow & \mathsf{LKSob} \\
\vee & & \vee & & \vee \\
\mathsf{StCFrm} & \leftrightsquigarrow & \mathsf{StCLPries} & \longleftrightarrow & \mathsf{StLKSp} \\
\vee & & \vee & & \vee \\
\mathsf{StKFrm} & \leftrightsquigarrow & \mathsf{StKLPries} & \longleftrightarrow & \mathsf{StKSp} \\
\vee & & \vee & & \vee \\
\mathsf{KRFrm} & \leftrightsquigarrow & \mathsf{KRLPries} & \longleftrightarrow & \mathsf{KHaus}
\end{array}
$$

Putting Theorems 7.17 and 7.18(2) together yields:

Corollary 7.20 KRLPries *is a full subcategory of* StKLPries.

We thus arrive at the diagram in Fig. 2, where we use the same notation as in Fig. 1. In addition, the arrow ($\longleftrightarrow$) represents an equivalence of categories. For an overview of the introduced categories of Priestley spaces see Table 3. The corresponding categories of frames and spaces are described in Tables 1 and 2.

We conclude the paper by proving that every L-morphism $f : X_1 \to X_2$ from an L-compact L-space to an L-regular L-space satisfies ${\downarrow} f(x) = f({\downarrow} x)$ for each $x \in X_1$. This result was first proved in [9, Cor. 4.3] utilizing that each frame homomorphism from a regular frame to a compact frame is closed (see [9, Lem. 4.1]). We give alternative proofs of both Lemma 4.1 and Corollary 4.3 of [9] using the language of Priestley spaces. We start by the following two lemmas.

Lemma 7.21 ([9, Rem. 3.7]) *Let X be an L-regular L-space and Y its localic part. Then* $\min(D) = \min(D')$ *implies $D = D'$ for all clopen downsets D, D' of X.*

Lemma 7.22 *Let X_1, X_2 be L-spaces and Y_1, Y_2 their respective localic parts. If $f : X_1 \to X_2$ is an L-morphism and D is a clopen downset of X_1, then*

(1) *${\downarrow} f(D)$ is clopen.*

If in addition X_1 is L-compact and X_2 is L-regular, then

(2) $\min {\downarrow} f(D) \subseteq f(D) \cap Y_2$.
(3) ${\downarrow} f(D) = f(D)$.
(4) *$f(D)$ is a clopen downset.*

Proof (1) Since f is a closed map, ${\downarrow} f(D)$ is closed, and hence $U := ({\downarrow} f(D))^c$ is an open upset. Because $D \cap f^{-1}(U) = \varnothing$ and f is an L-morphism,

$$D \cap f^{-1}(\mathsf{cl}\, U) = D \cap \mathsf{cl}\, f^{-1}(U) = \varnothing.$$

Therefore, $f(D) \cap \mathsf{cl}\, U = \varnothing$, and so ${\downarrow} f(D) \cap \mathsf{cl}\, U = \varnothing$ since $\mathsf{cl}\, U$ is an upset. Consequently, ${\downarrow} f(D) = (\mathsf{cl}\, U)^c$, and hence ${\downarrow} f(D)$ is clopen (because $\mathsf{cl}\, U$ is clopen).

(2) Suppose $z \in \min {\downarrow} f(D)$. Then $z \leq f(x)$ for some $x \in D$. Since D is closed, there is $y \in \min D$ with $y \leq x$ (see Lemma 3.4(5)). Because X_1 is L-compact, $\min D \subseteq \min X_1 \subseteq Y_1$. Therefore, $y \in Y_1$, and so $f(y) \in Y_2$ by Lemma 4.12(1). But then ${\uparrow} f(y)$ is a Scott upset. Thus, since X_2 is L-regular, ${\uparrow} f(y)$ is a biset by Lemma 7.15(2). Consequently, $f(x) \in {\uparrow} f(y)$ implies ${\downarrow} f(x) \subseteq {\uparrow} f(y)$. Therefore, $z \in {\uparrow} f(y)$, so $f(y) \leq z$. But then $f(y) = z$ by the minimality of z. Thus, $z \in f(D) \cap Y_2$.

Table 3 Categories of L-spaces

Category	Objects	Morphisms
LPries	L-spaces (Def. 3.8)	L-morphisms (Def. 3.8)
SLPries	L-spatial L-spaces (Def. 4.9)	L-morphisms
ConLPries	Continuous L-spaces (Def. 5.4)	Proper L-morphisms (Def. 5.12)
StCLPries	Stably continuous L-spaces (Def. 6.5)	Proper L-morphisms
StKLPries	Stably L-compact L-spaces (Def. 6.12)	Proper L-morphisms
KRLPries	L-compact L-regular L-spaces (Def. 7.6)	L-morphisms

(3) Clearly $f(D) \subseteq {\downarrow} f(D)$. To see the reverse inclusion, since $f(D)$ is a closed subset of X_2, it is sufficient to show that ${\downarrow} f(D)$ is the closure of $\min {\downarrow} f(D)$ because the latter is contained in $f(D)$ by (2). Suppose otherwise. Since ${\downarrow} f(D)$ is clopen by (1), ${\downarrow} f(D) \setminus \mathsf{cl} \min {\downarrow} f(D)$ is a nonempty open set. Therefore, by Lemma 3.4(1), there are $U, V \in \mathsf{ClopUp}(X_2)$ such that $\varnothing \neq U \cap V^c \subseteq {\downarrow} f(D)$ and $U \cap V^c \cap \mathsf{cl} \min {\downarrow} f(D) = \varnothing$. Let $A = {\downarrow}(U \cap V^c)$ and $B = A \setminus (U \cap V^c)$. Then $A \neq B$. Clearly A is a clopen downset and B is clopen. We show that B is also a downset and $\min A = \min B$. We first show that B is a downset. Let $x \in B$ and $y \leq x$. If $y \notin B$ then $y \in U \cap V^c$. Since $x \in {\downarrow}(U \cap V^c)$, there exists $z \in U \cap V^c$ such that $x \leq z$. Because U is an upset, from $y \in U$ and $y \leq x$ it follows that $x \in U$. Since V^c is a downset, from $z \in V^c$ and $x \leq z$ it follows that $x \in V^c$. Therefore, $x \in U \cap V^c$, and so $x \notin B$, a contradiction. Thus, B is a downset.

We next show that $\min A = \min B$. Clearly $\min B \subseteq \min A$. If $x \in \min A$, then $x \in \min {\downarrow} f(D)$, so $x \notin U$. Therefore, $x \in \min B$. Thus, A and B are clopen downsets such that $A \neq B$ but $\min A = \min B$. This contradicts Lemma 7.21 because X_2 is L-regular.

(4) Apply (1) and (3). □

Consequently, the same argument as in the proof of [9, Cor. 4.3] yields:

Theorem 7.23 *Let $f : X_1 \to X_2$ be a proper L-morphism between an L-compact L-space X_1 and an L-regular L-space X_2. Then $f({\downarrow} x) = {\downarrow} f(x)$ for each $x \in X_1$.*

We recall that a frame homomorphism $h : L \to M$ is *closed* if $r(h(a) \vee b) \leq a \vee r(b)$ for all $a \in L$ and $b \in M$, where $r : M \to L$ is the right adjoint of h. We close by an alternative proof of [9, Lem. 4.1].

Theorem 7.24 *If $h : L \to M$ is a frame homomorphism from a regular frame L to a compact frame M, then h is closed.*

Proof Let $f : X_M \to X_L$ be the dual L-morphism between the Priestley spaces of M and L, respectively. Suppose $a \in L$ and $b \in M$. Since X_L is L-regular by Lemma 7.7(1), it suffices to show that

$$\operatorname{reg} r(h(a) \vee b) \subseteq \varphi(a) \cup \varphi(r(b)).$$

Let $d \in M$. Since r is right adjoint to h, we have $r(d) = \bigvee \{c \in L \mid h(c) \leq d\}$. Therefore, since $\varphi(h(c)) = f^{-1}(\varphi(c))$, by Lemma 3.6 we have

$$\begin{aligned} \varphi(r(d)) &= \varphi\left(\bigvee \{c \in L \mid h(c) \leq d\}\right) \\ &= \mathsf{cl}\left(\bigcup \{\varphi(c) \mid f^{-1}(\varphi(c)) \subseteq \varphi(d)\}\right) \end{aligned}$$

$$= \mathsf{cl}\left(\bigcup\{\varphi(c) \mid \varphi(c) \subseteq X_L \setminus f(X_M \setminus \varphi(d))\}\right)$$
$$= X_L \setminus f(X_M \setminus \varphi(d)),$$

where the last equality follows from Lemma 7.22(4). Let $x \in \operatorname{reg} r(h(a) \vee b)$. We show that $x \in \varphi(a) \cup \varphi(r(b))$. By Lemma 7.3(1),

$$x \in \operatorname{reg} r(h(a) \vee b) \iff {\downarrow}{\uparrow} x \subseteq X_L \setminus f\big(X_M \setminus (f^{-1}\varphi(a) \cup \varphi(b))\big)$$
$$\iff f^{-1}({\downarrow}{\uparrow} x) \subseteq f^{-1}\varphi(a) \cup \varphi(b).$$

Suppose $x \notin \varphi(r(b))$. Then $x \notin X_L \setminus f(X_M \setminus \varphi(b))$, so $f^{-1}(x) \nsubseteq \varphi(b)$. Therefore, there is $z \notin \varphi(b)$ such that $f(z) = x$. From $f(z) = x$ it follows that

$$z \in f^{-1}({\downarrow}{\uparrow} x) \subseteq f^{-1}\varphi(a) \cup \varphi(b).$$

Hence, from $z \notin \varphi(b)$ it follows that $z \in f^{-1}\varphi(a)$, and so $x = f(z) \in \varphi(a)$, concluding the proof. □

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Conflict of interest Both authors declare that they have no conflicts of interest.

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