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Maximal d-spectra via Priestley duality

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We use Priestley duality as a new tool to study maximal d-spectra of arithmetic frames, both with and without units. We pay special attention to when the maximal d-spectrum is compact or Hausdorff. Various necessary and sufficient conditions are given, including a construction of an arithmetic frame with a unit whose maximal d-spectrum is not Hausdorff, thus resolving an open problem in the literature.

Keywords: Pointfree topology; Priestley duality; arithmetic frame; d-nucleus; maximal d-spectrum; compactness; Hausdorffness.

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1. Introduction

The space of maximal d-ideals of an archimedean Riesz space with a weak order unit, equipped with the hull-kernel topology, has been well studied and is known to be a compact Hausdorff space (see, e.g. [23]). Motivated by this, Martinez and Zenk [28] initiated the study of d-elements in an arbitrary arithmetic frame. These elements, denoted dL or L_d, form a sublocale of the arithmetic frame L. The corresponding

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nucleus was coined the *d-nucleus*. The frame L_d and the spectrum $\max L_d$ of maximal d-elements were further studied by various authors (see, e.g. [13–16]). It is known that $\max L_d$ is a compact T_1-space provided L has a unit. In [13], it was left open whether $\max L_d$ is Hausdorff. Although some characterizations of the Hausdorff separation for $\max L_d$ were recently established in [14], the question remained open. Our aim is to answer it in the negative.

Our main tool is Priestley duality for frames. Priestley originally developed her duality for bounded distributive lattices [33, 34]. It was restricted to the category of frames by Pultr and Sichler [36], and later the Priestley spaces of arithmetic frames were characterized in [11]. Building on this work, we describe the subset N_d of the Priestley space X of an arithmetic frame L corresponding to the d-nucleus on L. We also describe the subset Y_d of N_d corresponding to the spectrum $pt(L_d)$ of points of the sublocale L_d. We show that the minimum of Y_d is in a one-to-one correspondence with the maximal d-elements of L, thus yielding a homeomorphism between $\min Y_d$ and $\max L_d$. This allows us to study $\max L_d$ in the language of Priestley spaces. Our main results include the description of the nucleus on L whose fixpoints are the frame of opens of $\min Y_d$, the characterization of the soberification of $\min Y_d$, and the construction of an arithmetic frame L with a unit such that $\min Y_d$ is not Hausdorff. This yields that $\max L_d$ is not Hausdorff, thus resolving the open question mentioned above. We also give a necessary and sufficient condition for $\max L_d$ to be Hausdorff. This characterization remains valid even if L doesn't have a unit, provided $\max L_d$ is locally compact. We also investigate the compactness of $\min Y_d$, and hence of $\max L_d$, in comparison with the existence of a unit. Our approach raises new open questions and highlights the need for further study of maximal d-spectra of arithmetic frames (see the end of the paper).

The paper is structured as follows. In Sec. 2, we recall Priestley duality for arithmetic frames, along with useful definitions and results from the literature. In Sec. 3, we revisit the relationship between nuclei and sublocales, as well as their description in the language of Priestley spaces. In particular, we show how to use Priestley duality to give alternative proofs of two existing results in the literature; Johnstone's lemma that each Scott open filter arises as the dense elements of a nucleus and the Isbell Density Theorem. In Sec. 4, we characterize inductive nuclei in the language of Priestley spaces, as well as provide the dual description of the d-nucleus on an arithmetic frame L. In Sec. 5, we study the maximum of the localic part of the Priestley dual of L, which yields a new characterization of when the sublocale L_d is regular. In Sec. 6, we delve into the investigation of the spectrum $\max L_d$ of maximal d-elements of an arithmetic frame. We establish a homeomorphism between $\max L_d$ and $\min Y_d$, thus providing us with a new tool to study the maximal d-spectrum of L. We show that the frame of open sets of $\min Y_d$ can be realized as a sublocale of L and describe the corresponding nuclear subset of X. In addition, we prove that the localic part of this nuclear subset is the soberification of $\min Y_d$.

In Sec. 7, we study the topological properties of $\min Y_d$ in comparison to what is known about $\max L_d$. We describe compact subsets of $\min Y_d$, which allows us to characterize when an arithmetic frame has a unit using Priestley duality. This in particular yields that in the presence of a unit, $\min Y_d$ is a compact space.

Finally, in Sec. 8, we explore the Hausdorff separation for the space $\min Y_d$. An example of an arithmetic frame L with a unit is constructed such that $\max L_d$ and hence $\min Y_d$ is not Hausdorff, thus resolving an open question from [13]. Furthermore, we give a characterization of when $\min Y_d$ is Hausdorff, which generalizes to arithmetic frames without units provided $\min Y_d$ is locally compact. The paper concludes with several open questions that the authors are looking into to further the study of $\min Y_d$.

2. Priestley Duality for Frames

A *frame* is a complete lattice L such that

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$. A *frame homomorphism* is a map between frames that preserves finite meets and arbitrary joins. Let Frm be the category of frames and frame homomorphisms. A frame L is *spatial* if it is isomorphic to the frame of opens of a topological space (equivalently, completely prime filters separate elements of L). Let SFrm be the full subcategory of Frm consisting of spatial frames.

An element $a \in L$ is *compact* if for each $S \subseteq L$, from $a \leq \bigvee S$ it follows that $a \leq \bigvee T$ for some finite $T \subseteq S$. Let $K(L)$ be the set of compact elements of L. Then L is *algebraic* provided

$$a = \bigvee \{b \in K(L) \mid b \leq a\}$$

for each $a \in L$. It is well known (see, e.g. [28, Remark 3.4]) that every algebraic frame is spatial. A frame homomorphism $h : L \to M$ is *coherent* if $h[K(L)] \subseteq K(M)$. Let AlgFrm be the category of algebraic frames and coherent frame homomorphisms between them. An algebraic frame L is *arithmetic* if $K(L)$ is closed under binary meets. Let AriFrm be the full subcategory of AlgFrm consisting of arithmetic frames.

A space is *zero-dimensional* if it has a basis of clopen sets. A *Stone space* is a zero-dimensional, compact, Hausdorff space. A *Priestley space* is a pair $(X, \leq)$ such that X is a Stone space and $\leq$ is a partial order on X such that the *Priestley separation* holds:

If $x \nleq y$ then there is a clopen upset U of X containing x and missing y.

A *Priestley morphism* is a continuous order-preserving map between Priestley spaces. Let Pries be the category of Priestley spaces and Priestley morphisms. Let DLat be the category of bounded distributive lattice and bounded lattice homomorphisms.

Theorem 2.1 (Priestley duality). *DLat and Pries are dually equivalent.*

Remark 2.2. The functors $\mathscr{X} : \mathsf{DLat} \to \mathsf{Pries}$ and $\mathscr{D} : \mathsf{Pries} \to \mathsf{DLat}$ establishing Priestley duality are described as follows:

- The *Priestley space* of a bounded distributive lattice D is the set X_D of prime filters of D ordered by inclusion and topologized by the basis $\{\varphi(a)\backslash\varphi(b) \mid a, b \in D\}$, where φ is the *Stone map* defined by $\varphi(a) = \{x \in X_D \mid a \in x\}$ for all $a \in D$. The functor $\mathscr{X}$ assigns to each $D \in \mathsf{DLat}$ its Priestley space X_D, and to each bounded lattice homomorphism $h : D \to E$ the Priestley morphism $h^{-1} : X_E \to X_D$.
- The functor $\mathscr{D}$ assigns to each Priestley space X the bounded distributive lattice $\mathsf{ClopUp}(X)$ of clopen upsets of X and to each Priestley morphism $f : X \to Y$ the bounded lattice homomorphism $f^{-1} : \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$.

Definition 2.3. (1) An *L-space* (*localic space*) is a Priestley space such that the closure of each open upset is an open upset.
(2) An *L-morphism* is a Priestley morphism $f : X \to Y$ between L-spaces satisfying $f^{-1}(\mathrm{cl}\, U) = \mathrm{cl}\, f^{-1}(U)$ for each open upset U of Y.
(3) LPries is the category of L-spaces and L-morphisms.

Remark 2.4. Recall that frames are precisely complete Heyting algebras. By Esakia duality [19], the Priestley spaces of Heyting algebras (Esakia spaces) are those with the property that the closure of each upset is an upset. Moreover, if X is the Priestley space of a Heyting algebra L, then L is complete iff X is *extremally order-disconnected*; that is, the closure of each open upset is open. Thus, L-spaces are precisely extremally order-disconnected Esakia spaces.

Theorem 2.5 (Pultr-Sichler duality). Frm *and* LPries *are dually equivalent.*

Throughout we will use the following well-known facts (see, e.g. [3, 20, 35]).

Lemma 2.6. *Let X be a Priestley space.*

(1) *If $F \subseteq X$ is closed, then so are $\uparrow F$ and $\downarrow F$.*
(2) *If $F \subseteq X$ is a closed upset, then it is an intersection of clopen upsets.*
(3) *If $F \subseteq X$ is closed, then for each $x \in F$ there are $m \in \min F$ and $n \in \max F$ such that $m \le x \le n$, where $\min F$ and $\max F$ denote the sets of minimal and maximal points of F, respectively.*

If in addition X is an L-space, then we have:

(4) *If $F \subseteq X$ is closed, then so is $\max F$.*
(5) *If $U \subseteq X$ is clopen, then so is $\downarrow U$.*
(6) *For $\{U_i\} \subseteq \mathsf{ClopUp}(X)$,*

$$\bigvee U_i = \mathrm{cl} \bigcup U_i \quad \text{and} \quad \bigwedge U_i = X \backslash {\downarrow}(X \backslash \mathrm{int} \bigcap U_i).$$

We next describe the Priestley spaces of spatial, algebraic, and arithmetic frames.

Definition 2.7. For an L-space X,

(1) $y \in X$ is a *localic point* if $\downarrow y$ is clopen;
(2) the set Y of localic points of X is the *localic part* of X;
(3) X is an *SL-space* (*spatial* L-space) if Y is dense in X;
(4) SLPries is the full subcategory of LPries consisting of SL-spaces.

Theorem 2.8 (see, e.g. [10, Corollary 4.10]). SFrm *and* SLPries *are dually equivalent.*

Remark 2.9. Let X be the Priestley space of a frame L, and Y the localic part of X. We topologize Y by $\{U \cap Y \mid U \in \mathsf{ClopUp}(X)\}$. Then Y is precisely the space of points of L (see, e.g. [1, Proposition 5.1]).

Definition 2.10. Let X be an L-space and Y its localic part.

(1) A closed upset $F \subseteq X$ is a *Scott upset* if $\min F \subseteq Y$.
(2) $\mathsf{ClopSUp}(X)$ is the collection of clopen Scott upsets of X.

Scott upsets play an important role in Priestley spaces of frames as they correspond to Scott open filters on the frame side.

Theorem 2.11 ([9, Theorem 5.6]). *Let L be a frame and X its L-space. The poset of Scott open filters of L is dually isomorphic to the poset of Scott upsets of X.*

Definition 2.12. Let X be an L-space and Y its localic part.

(1) The *core* of $U \in \mathsf{ClopUp}(X)$ is $\operatorname{core} U = \bigcup\{V \in \mathsf{ClopSUp}(X) \mid V \subseteq U\}$.
(2) X is an *algebraic L-space* if $\operatorname{core} U$ is dense in U for each $U \in \mathsf{ClopUp}(X)$.
(3) An L-morphism $f : X_1 \to X_2$ is *coherent* if $f^{-1}(\operatorname{core} U) \subseteq \operatorname{core} f^{-1}(U)$ for each $U \in \mathsf{ClopUp}(X_2)$.
(4) AlgL is the category of algebraic L-spaces and coherent L-morphisms between them.

Theorem 2.13 ([11, Theorem 4.9]). AlgFrm *and* AlgL *are dually equivalent.*

Consequently, every algebraic L-space is an SL-space.

Remark 2.14. Let X be an L-space and $U \in \mathsf{ClopUp}(X)$. Then $\operatorname{core} U = U$ iff $U \in \mathsf{ClopSUp}(X)$, which by [9, Corollary 5.4] is equivalent to U being a compact element of $\mathsf{ClopUp}(X)$. Thus, $\mathsf{ClopSUp}(X) = K(\mathsf{ClopUp}(X))$.

Definition 2.15. (1) An *arithmetic L-space* is an algebraic L-space X such that

$$\operatorname{core} U \cap \operatorname{core} V = \operatorname{core}(U \cap V)$$

for all $U, V \in \mathsf{ClopUp}(X)$. Equivalently (see, e.g. [11, Lemma 5.2]),

$$U, V \in \mathsf{ClopSUp}(X) \Rightarrow U \cap V \in \mathsf{ClopSUp}(X).$$

(2) AriL is the full subcategory of AlgL consisting of arithmetic L-spaces.

Theorem 2.16 ([11, Theorem 5.5]). AriFrm *and* AriL *are dually equivalent.*

3. Priestley Duals for Nuclei and Sublocales

In this section, we recall the relationship between nuclei and sublocales of a frame L, as well as their dual characterization as nuclear subsets of the Priestley space of L. Moreover, we give a dual characterization of the admissible filter of a nucleus on L, which yields an alternative proof of [26, Lemma 3.4(ii)]. Furthermore, we provide an alternate proof of Isbell's Density Theorem, which utilizes Priestley duality.

Definition 3.1 (see, e.g. [25, p. 48]). A *nucleus* on a frame L is a map $j : L \to L$ satisfying

(1) $a \le ja$,
(2) $jja \le ja$,
(3) $ja \wedge jb = j(a \wedge b)$

for all $a, b \in L$.

Let $N(L)$ be the set of nuclei on L. We order $N(L)$ pointwise, i.e. $j \le k$ iff $j(a) \le k(a)$ for all $a \in L$. With this order, it is well known that $N(L)$ is a frame (see, e.g. [25, p. 51]).

Definition 3.2 (see, e.g. [31, p. 26]). Let L be a frame. We call $S \subseteq L$ a *sublocale* of L provided S is closed under arbitrary meets and $a \to s \in S$ for all $a \in L$ and $s \in S$.

Let $S(L)$ be the set of sublocales of L. We order $S(L)$ by inclusion. Then $S(L)$ is dually isomorphic to $N(L)$ (see, e.g. [31, Sec. III-5]). The dual isomorphism associates with each nucleus j, the sublocale $S_j := j[L] \in S(L)$; and with each sublocale S, the nucleus j_S given by $j_S(a) = \bigwedge\{s \in S \mid a \le s\}$. Nuclei, and hence sublocales, dually correspond to nuclear subsets of L-spaces introduced in [37] (see also [1, 7]):

Definition 3.3. Let X be an L-space.

(1) A subset N of X is a *nuclear subset* provided N is closed and $\downarrow(U \cap N)$ is clopen for each clopen subset U of X.
(2) Let $N(X)$ be the set of nuclear subsets of X ordered by inclusion.

Theorem 3.4 ([7, Theorem 30]). *Let L be a frame and X its Priestley space. Then $N(L)$ is dually isomorphic to $N(X)$.*

Remark 3.5. The dual isomorphism of the previous theorem is established as follows: with each $j \in N(L)$ we associate the nuclear subset

$$N_j := \{x \in X \mid j^{-1}[x] = x\} \in N(X),$$

and with each $N \in N(X)$ the nucleus $j_N \in N(\mathsf{ClopUp}(X))$ given by $j_N U = X \setminus \downarrow(N \setminus U)$. Then $j := \varphi^{-1} \circ j_N \circ \varphi$ is the corresponding nucleus on L.

To simplify notation, we identify $N(L)$ with $N(\mathsf{ClopUp}(X))$. Thus, each $j \in N(L)$ is identified with j_{N_j}, and hence, for $U \in \mathsf{ClopUp}(X)$, we have $j\,U = X \setminus {\downarrow}(N_j \setminus U)$.

Corollary 3.6 ([37, p. 229]). *Let L be a frame and X its Priestley space. Then $S(L)$ is isomorphic to $N(X)$.*

Remark 3.7. Let L be a frame and X its Priestley space.

(1) If $j \in N(L)$, then N_j seen as a subspace of X is order-homeomorphic to the Priestley space of the sublocale S_j of L (see, e.g. [7, Lemma 25]).
(2) Localic points of X are also known as *nuclear points* (see, e.g. [1, Definition 4.1]). This is because $y \in X$ is localic iff $\{y\}$ is a nuclear subset.
(3) By [1, Lemma 4.8], the join in $N(X)$ is calculated by

$$\bigvee N_i = \mathrm{cl} \bigcup N_i$$

for $\{N_i\} \subseteq N(X)$.
(4) By (3), $\mathrm{cl}\, Z$ is a nuclear subset of X for every subset $Z \subseteq Y$ of the localic part.

Let j be a nucleus on L. An element $a \in L$ is called *j-dense* provided $ja = 1$. Let F_j be the set of all j-dense elements of L. It is well known and straightforward to verify that F_j is a filter of L.

Definition 3.8 (see, e.g. [39]). A filter F of a frame L is called *admissible* if it is of the form $F = F_j$ for some $j \in N(L)$.

Remark 3.9. In [27] admissible filters are called *smooth*. In [40, Theorem 25.5] it is shown that a filter is admissible iff it is *free*, a concept that is now known as *strongly exact* (see, e.g. [29]).

Let X be the Priestley space of L. Recalling the well-known correspondence between filters of L and closed upsets of X (see [35, p. 54] or [5, Corollary 6.3]), let $H_j := \bigcap \varphi[F_j]$ be the closed upset of X corresponding to F_j. To describe the relationship between N_j and H_j, we require the following lemma.

Lemma 3.10. *Let L be a frame, X its Priestley space, $j \in N(L)$, and $a, b \in L$.*

(1) $\varphi(a) \cap N_j = \varphi(ja) \cap N_j$.
(2) $x \in \varphi(ja)$ *iff* ${\uparrow}x \cap N_j \subseteq \varphi(a) \cap N_j$.
(3) $ja \leq jb$ *iff* $\varphi(a) \cap N_j \subseteq \varphi(b) \cap N_j$.
(4) $ja = jb$ *iff* $\varphi(a) \cap N_j = \varphi(b) \cap N_j$.
(5) *a is j-dense iff* $N_j \subseteq \varphi(a)$.

Proof. (1) The left-to-right inclusion is clear because $a \leq ja$. For the right-to-left inclusion, suppose $x \in \varphi(ja) \cap N_j$. Then $ja \in x$, so $a \in j^{-1}[x]$. But $j^{-1}[x] = x$ since $x \in N_j$. Consequently, $x \in \varphi(a) \cap N_j$.

(2) Let $x \in X$. We have

$$\begin{aligned} x \in \varphi(ja) &\Leftrightarrow x \in X \backslash {\downarrow}(N_j \backslash \varphi(a)) \\ &\Leftrightarrow x \notin {\downarrow}(N_j \backslash \varphi(a)) \\ &\Leftrightarrow {\uparrow}x \cap (N_j \backslash \varphi(a)) = \varnothing \\ &\Leftrightarrow {\uparrow}x \cap N_j \subseteq \varphi(a) \cap N_j. \end{aligned}$$

(3) Suppose $ja \leq jb$. Then $\varphi(ja) \subseteq \varphi(jb)$. Consequently,

$$\varphi(ja) \cap N_j \subseteq \varphi(jb) \cap N_j$$

and so $\varphi(a) \cap N_j \subseteq \varphi(b) \cap N_j$ by (1). Conversely, suppose $\varphi(a) \cap N_j \subseteq \varphi(b) \cap N_j$. It suffices to show that $\varphi(ja) \subseteq \varphi(jb)$. Let $x \in \varphi(ja)$. Then ${\uparrow}x \cap N_j \subseteq \varphi(a) \cap N_j$ by (2). Therefore, ${\uparrow}x \cap N_j \subseteq \varphi(b) \cap N_j$ by assumption. Thus, $x \in \varphi(jb)$ by (2).

(4) This follows from (3).

(5) Suppose $ja = 1$. Then $\varphi(ja) = X$. Therefore, by (1),

$$N_j \cap \varphi(a) = N_j \cap \varphi(ja) = N_j.$$

Thus, $N_j \subseteq \varphi(a)$. Conversely, suppose $N_j \subseteq \varphi(a)$. Then $\varphi(a) \cap N_j = N_j = \varphi(1) \cap N_j$. Therefore, by (4), $ja = j1 = 1$. □

Theorem 3.11. *Let L be a frame, X its Priestley space, and $j \in N(L)$. Then $H_j = {\uparrow}N_j$.*

Proof. Since both H_j and ${\uparrow}N_j$ are closed upsets, and hence intersections of clopen upsets (see Lemma 2.6(2)), it is sufficient to show that $H_j \subseteq \varphi(a)$ iff ${\uparrow}N_j \subseteq \varphi(a)$ for each $a \in L$. We have

$$\begin{aligned} H_j = \bigcap_{b \in F_j} \varphi(b) \subseteq \varphi(a) \Leftrightarrow a \in F_j \Leftrightarrow ja = 1 \\ \Leftrightarrow N_j \subseteq \varphi(a) \Leftrightarrow {\uparrow}N_j \subseteq \varphi(a), \end{aligned}$$

where the first equivalence follows from compactness and the second to last equivalence from Lemma 3.10(5). □

The following result about Scott open filters was first established in [26, Lemma 3.4(ii)] using transfinite induction. Since then various alternative proofs have been obtained (see, e.g. [24, Sec. 5.1] and the references therein). We utilize Priestley duality to provide yet another alternative proof.

Corollary 3.12. *Every Scott open filter of a frame is admissible.*

Proof. Let L be a frame and X its L-space. By Theorem 2.11, Scott open filters correspond to Scott upsets; and by Theorem 3.11, admissible filters correspond to closed upsets of the form ${\uparrow}N$ for some $N \in N(X)$. Therefore, it suffices to show that for each Scott upset F, there exists a nuclear subset $N \subseteq X$ such that $F = \uparrow N$. Let $N = \mathrm{cl}(F \cap Y)$. Then N is nuclear by Remark 3.7(4). Moreover, since F is a Scott upset, $\min F \subseteq Y$, and hence $F = \uparrow \min F = \uparrow \mathrm{cl}(F \cap Y) = \uparrow N$. □

Remark 3.13. Our proof that Scott open filters are admissible relies on Lemma 2.6(3), which requires the Axiom of Choice. An alternative proof using only the Prime Ideal Theorem can be found in [12, Remark 5.8].

One of the most studied nuclei is the nucleus of double-negation. For a frame L and $a \in L$, recall that the *pseudocomplement* of a is given by

$$a^* = \bigvee \{b \in L \mid b \wedge a = 0\}.$$

The map $a \mapsto a^{**}$ is the *double-negation nucleus*, and the corresponding sublocale

$$\mathfrak{B}(L) := \{a \in L \mid a = a^{**}\}$$

is the *Booleanization* of L (see, e.g. [32, p. 246]).

If j is the double-negation nucleus, then j-dense elements are simply called *dense* (see, e.g. [38, p. 131]). It is well known that the corresponding admissible filter dually corresponds to $\max X$, and so we have:

Proposition 3.14. *Let j be the double-negation nucleus on L.*

(1) $H_j = \max X$.
(2) $N_j = \max X$.

Proof. For (1) see, e.g. [2, Sec. 3]. For (2) observe that (1) and Theorem 3.11 yield $\max X = H_j = \uparrow N_j$. Consequently, $N_j = \max X$. □

The following definition is well known. For parts (1) and (2) see, e.g. [25, p. 50] and for part (3) see, e.g. [4, p. 108].

Definition 3.15. Let L be a frame and X its Priestley space.

(1) $j \in N(L)$ is *dense* if $j0 = 0$.
(2) $S \in S(L)$ is *dense* if $0 \in S$.
(3) $N \in N(X)$ is *cofinal* if $\max X \subseteq N$.

Lemma 3.16. *Let L be a frame, X its Priestley space, and $j \in N(L)$. The following are equivalent.*

(1) *j is dense.*
(2) *S_j is dense.*
(3) *N_j is cofinal.*

Proof. The equivalence (1)⇔(2) is obvious, and (1)⇔(3) is proved in [7, Theorems 23(2) and 28(2)]. □

Theorem 3.17. *Let X be an L-space. Then $\max X$ is the least cofinal nuclear subset.*

Proof. By Proposition 3.14(2), $\max X$ is a nuclear subset of X, and clearly it is the least such containing $\max X$. Thus, it is the least cofinal nuclear subset of X. □

As a consequence, we obtain the following well-known result of Isbell (see, e.g. [31, p. 40]):

Corollary 3.18 (Isbell's Density Theorem). *For a frame L, the Booleanization $\mathfrak{B}(L)$ is the least dense sublocale of L.*

Proof. Let $S_j \subseteq L$ be a dense sublocale. Therefore, N_j is cofinal by Lemma 3.16, and so $\max X \subseteq N_j$ by Theorem 3.17. Consequently, $\mathfrak{B}(L) \subseteq S_j$ by Corollary 3.6 and Proposition 3.14(2). □

4. Priestley Duality for the *d*-Nucleus

In this section, we describe Priestley duals of inductive nuclei, introduced and studied by Martinez and Zenk [28]. We use the Priestley duality tools from previous sections to study the most prominent inductive nucleus, known as the d-nucleus. Among other things, we characterize the nuclear set N_d corresponding to the d-nucleus, and its localic part Y_d.

Definition 4.1 ([28, Sec. 4]). A nucleus j on an algebraic frame L is *inductive* if for all $a \in L$ we have

$$ja = \bigvee\{jk \mid k \in K(L) \text{ and } k \leq a\}.$$

Let X be the Priestley space of L. As we pointed out after Remark 3.5, we identify $N(L)$ with $N(\mathsf{ClopUp}(X))$, so for each $j \in N(L)$ there is a unique $N \in N(X)$ such that $j\,U = X\backslash{\downarrow}(N\backslash U)$ for each $U \in \mathsf{ClopUp}(X)$.

Definition 4.2. Let X be an L-space. For $j \in N(\mathsf{ClopUp}(X))$ and $U \in \mathsf{ClopUp}(X)$, define the *j-core of U* by

$$\text{core}_j U = \bigcup\{j\,V \mid V \in \mathsf{ClopSUp}(X) \text{ and } V \subseteq U\}.$$

Remark 4.3. Let j be a nucleus on an arithmetic frame L and let X be the Priestley space of L. Then X is an arithmetic L-space by Theorem 2.16. Therefore, for all clopen upsets U, V of X,

$$\begin{aligned}\text{core}_j\, U \cap \text{core}_j\, V &= \bigcup\{j\,U' \mid U' \in \mathsf{ClopSUp}(X),\ U' \subseteq U\}\\ &\quad \cap \bigcup\{j\,V' \mid V' \in \mathsf{ClopSUp}(X),\ V' \subseteq U\}\\ &= \bigcup\{j\,(U' \cap V') \mid U', V' \in \mathsf{ClopSUp}(X),\ U' \subseteq U,\ V' \subseteq V\}\\ &= \bigcup\{jW \mid W \in \mathsf{ClopSUp}(X),\ W \subseteq U \cap V\}\\ &= \text{core}_j(U \cap V),\end{aligned}$$

where the second to last equality is a consequence of X being an arithmetic L-space (see Definition 2.15(1)).

Definition 4.4. Let X be an L-space. We call $N \in N(X)$ *inductive* if $\uparrow(F \cap N)$ is a Scott upset for each Scott upset F of X.

As the name suggests, a nuclear subset is inductive iff its corresponding nucleus is inductive. To prove this, we recall the following:

Lemma 4.5. *Let X be an L-space and Y its localic part.*

(1) ([9, Lemma 5.1]) *A closed upset F of X is a Scott upset iff $F \subseteq \operatorname{cl} U$ implies $F \subseteq U$ for all open upsets U of X.*
(2) ([10, Lemma 4.14]) *Let $y \in Y$ and U be an open upset of X. Then $y \in U$ iff $y \in \operatorname{cl} U$.*

Theorem 4.6. *Let X be an algebraic L-space and $N \in N(X)$. The following are equivalent.*

(1) *N is inductive.*
(2) *$j_N U = \operatorname{cl} \operatorname{core}_{j_N} U$ for all $U \in \mathsf{ClopUp}(X)$.*
(3) *j_N is inductive.*

Proof. (1) $\Rightarrow$ (2) Let $U \in \mathsf{ClopUp}(X)$. Clearly, $\operatorname{cl} \operatorname{core}_{j_N} U \subseteq j_N U$. For the other inclusion, since Y is dense in X (every algebraic L-space is an SL-space), it is sufficient to show that $j_N(U) \cap Y \subseteq \operatorname{core}_{j_N} U$. Suppose that $y \in j_N(U) \cap Y$. Then $\uparrow y \cap N \subseteq U$ by Lemma 3.10(2). Since $\uparrow y$ is a Scott upset and N is inductive, $\uparrow(\uparrow y \cap N)$ is a Scott upset. But

$$\uparrow(\uparrow y \cap N) \subseteq U = \operatorname{cl} \operatorname{core} U$$

and hence $\uparrow(\uparrow y \cap N) \subseteq \operatorname{core} U$ by Lemma 4.5(1). Therefore, since finite unions of clopen Scott upsets are clopen Scott upsets, by compactness there exists $V \in \mathsf{ClopSUp}(X)$ such that $\uparrow y \cap N \subseteq V$ and $V \subseteq U$. Thus, $y \in j_N V$ by Lemma 3.10(2), and so $y \in \operatorname{core}_{j_N} U$, proving that $j_N(U) \cap Y \subseteq \operatorname{core}_{j_N} U$.

(2) $\Rightarrow$ (1) Let F be a Scott upset and $\uparrow(F \cap N) \subseteq \operatorname{cl} U$ for some open upset U. Since X is an L-space, $U' = \operatorname{cl} U \in \mathsf{ClopUp}(X)$. Let $y \in \min F$. Then

$$\uparrow y \cap N \subseteq \uparrow(F \cap N) \subseteq U',$$

so $y \in j_N U'$ by Lemma 3.10(2). Therefore, $y \in \operatorname{cl} \operatorname{core}_{j_N} U'$ by (2), and hence $y \in \operatorname{core}_{j_N} U'$ by Lemma 4.5(2). Thus, there is $V_y \in \mathsf{ClopSUp}(X)$ such that $y \in j_N V_y$ and $V_y \subseteq U'$. We have $F = \uparrow \min F \subseteq \bigcup \{j_N V_y \mid y \in \min F\}$, so by compactness there are $V_1, \dots, V_n \in \mathsf{ClopSUp}(X)$ such that $F \subseteq j_N V_1 \cup \cdots \cup j_N V_n$. Let $V = V_1 \cup \cdots \cup V_n$. Then $V \in \mathsf{ClopSUp}(X)$. Furthermore, $F \subseteq j_N V$ since j_N is order-preserving, and clearly $V \subseteq U'$. The latter together with Lemma 4.5(1) yields that $V \subseteq U$. Since $F \subseteq j_N V$, we have $F \cap N \subseteq j_N(V) \cap N = V \cap N$ by

Lemma 3.10(1). Consequently, $\uparrow(F \cap N) \subseteq U$, and hence $\uparrow(F \cap N)$ is a Scott upset by Lemma 4.5(1).

(2) ⇔ (3) Observe that

$$j_N \text{ is inductive} \Leftrightarrow j_N\, U = \bigvee\{j_N\, V \mid V \in K(\mathsf{ClopUp}(X)) \text{ and } V \leq U\}.$$

But $K(\mathsf{ClopUp}(X)) = \mathsf{ClopSUp}(X)$ (see Remark 2.14) and $\bigvee \mathcal{U} = \operatorname{cl} \bigcup \mathcal{U}$ for $\mathcal{U} \subseteq \mathsf{ClopUp}(X)$ (see Lemma 2.6(6)). Therefore,

$$\bigvee\{j_N\, V \mid V \in K(\mathsf{ClopUp}(X)) \text{ and } V \leq U\} = \operatorname{cl} \operatorname{core}_{j_N} U.$$

Consequently, j_N is inductive iff $U = \operatorname{cl} \operatorname{core}_{j_N} U$ for all $U \in \mathsf{ClopUp}(X)$. □

A prominent example of an inductive nucleus is the so-called d-nucleus, introduced by Martinez and Zenk [28, Sec. 5] as a frame-theoretic tool to study d-ideals of Riesz spaces (see [23; 28, Remark 5.6]).

Definition 4.7. Let L be an algebraic frame.

(1) Define $d : L \to L$ by $da = \bigvee\{k^{**} \mid k \in K(L) \text{ and } k \leq a\}$ for all $a \in L$.
(2) We call $a \in L$ a *d-element* if $da = a$.

We write L_d for the fixpoints of d.

Theorem 4.8 ([28, Sec. 5]). *Let L be an algebraic frame.*

(1) *d is a closure operator on L.*

If in addition L is arithmetic, then

(2) *d is an inductive dense nucleus on L.*
(3) *L_d is a dense sublocale of L and is an arithmetic frame.*

We next describe the nuclear subset N_d of the Priestley space of L corresponding to the sublocale L_d. As in Remark 3.7(1), for each nucleus j on a frame L, we view the corresponding nuclear subset N_j as the Priestley space of the sublocale L_j.

Proposition 4.9. *Let L be a frame, X its Priestley space, and Y the localic part of X.*

(1) *If $j \in N(L)$, then $N_j \cap Y$ is the localic part of N_j.*

If in addition L is an arithmetic frame, then

(2) *N_d is a cofinal inductive nuclear subset of X.*
(3) *N_d is an arithmetic L-space.*
(4) *$N_d = \operatorname{cl}(N_d \cap Y)$.*

Proof. (1) Let Y_j be the localic part of N_j. We need to show that $Y_j = N_j \cap Y$. It is straightforward to see that $N_j \cap Y \subseteq Y_j$. It remains to show that $Y_j \subseteq Y$.

Suppose $y \in Y_j$. Then $\downarrow y \cap N_j$ is clopen in N_j. Therefore, there is clopen $U \subseteq X$ such that $\downarrow y \cap N_j = U \cap N_j$. Since N_j is a nuclear set, $\downarrow(U \cap N_j)$ is clopen in X. But $\downarrow(U \cap N_j) = \downarrow(\downarrow y \cap N_j) = \downarrow y$ because $y \in N_j$. Thus, $y \in Y$.

(2) Apply Theorems 4.8(2) and 4.6, and Lemma 3.16.

(3) Since L is arithmetic, so is L_d by Theorem 4.8(3). Therefore, the result follows from Theorem 2.16 because N_d is the Priestley dual of L_d.

(4) $N_d \cap Y$ is the localic part of N_d by (1), and N_d is an SL-space by (3). Hence, $\mathrm{cl}(N_d \cap Y) = N_d$. □

Let X be an arithmetic L-space. Since $\mathsf{ClopSUp}(X) = K(\mathsf{ClopUp}(X))$ (see Remark 2.14), $d : \mathsf{ClopUp}(X) \to \mathsf{ClopUp}(X)$ is given by

$$dU = \mathrm{cl} \bigcup \{V^{**} \mid V \in \mathsf{ClopSUp}(X) \text{ and } V \subseteq U\},$$

where $V^* = X \backslash \downarrow V$ (see, e.g. [20, p. 20]), so $x \in V^{**}$ iff $\uparrow x \subseteq \downarrow V$. We also recall (see Definition 4.2) that the d-core of $U \in \mathsf{ClopUp}(X)$ is given by

$$\mathrm{core}_d\, U = \bigcup \{dV \mid V \in \mathsf{ClopSUp}(X) \text{ and } V \subseteq U\}.$$

Lemma 4.10. *Let L be an arithmetic frame, X its Priestley space, and Y the localic part of X.*

(1) *If $U \in \mathsf{ClopSUp}(X)$, then $dU = U^{**}$.*
(2) *If $U \in \mathsf{ClopUp}(X)$, then $x \in \mathrm{core}_d\, U$ iff $\uparrow x \subseteq \downarrow \mathrm{core}\, U$.*

Proof. (1) $dU = \mathrm{cl} \bigcup \{V^{**} \mid V \in \mathsf{ClopSUp}(X) \text{ and } V \subseteq U\} = U^{**}$ since $U \in \mathsf{ClopSUp}(X)$.

(2) First suppose that $x \in \mathrm{core}_d\, U$. Then there is $V \in \mathsf{ClopSUp}(X)$ with $x \in dV$ and $V \subseteq U$. Therefore, $x \in V^{**}$ by (1), which means that $\uparrow x \subseteq \downarrow V$. Since V is a Scott upset, $V \subseteq U$ implies $V \subseteq \mathrm{core}\, U$. Thus, $x \in dV$ implies $\uparrow x \subseteq \downarrow \mathrm{core}\, U$. For the converse, if $\uparrow x \subseteq \downarrow \mathrm{core}\, U$, then

$$\uparrow x \subseteq \downarrow \bigcup \{V \in \mathsf{ClopSUp}(X) \mid V \subseteq U\} = \bigcup \{\downarrow V \mid V \subseteq U \text{ and } V \in \mathsf{ClopSUp}(X)\}.$$

Hence, by Lemma 2.6(5), $\{\downarrow V \mid V \subseteq U \text{ and } V \in \mathsf{ClopSUp}(X)\}$ is an open cover of $\uparrow x$. Since $\uparrow x$ is compact and this open cover is directed, there is $V \in \mathsf{ClopSUp}(X)$ such that $V \subseteq U$ and $\uparrow x \subseteq \downarrow V$. This yields that $x \in V^{**}$, so $x \in dV$ by (1). Consequently, $x \in \mathrm{core}_d\, U$. □

Let X be an arithmetic L-space and Y its localic part. We let Y_d denote the localic part of N_d. By Proposition 4.9(1), $Y_d = N_d \cap Y$. We conclude the section by giving several characterizations of Y_d. For this we need the following lemma.

Lemma 4.11. *Let X be an algebraic L-space and F a Scott upset of X. Then*

$$F = \bigcap \{U \in \mathsf{ClopSUp}(X) \mid F \subseteq U\}.$$

Proof. Since F is a closed upset, $F = \bigcap\{U \in \mathsf{ClopUp}(X) \mid F \subseteq U\}$ (see Lemma 2.6(2)). Thus, it suffices to show that for each $U \in \mathsf{ClopUp}(X)$ with $F \subseteq U$, there is $V \in \mathsf{ClopSUp}(X)$ with $F \subseteq V \subseteq U$. Since X is an algebraic L-space, $U = \mathrm{cl}\,\mathrm{core}\,U$, so $F \subseteq U$ implies that $F \subseteq \mathrm{core}\,U$ by Lemma 4.5(1). Now apply compactness to obtain the desired V. □

Theorem 4.12. *Let L be an arithmetic frame, X its Priestley space, and Y the localic part of X. For $y \in Y$, the following are equivalent.*

(1) $y \in Y_d$.
(2) $\forall U \in \mathsf{ClopUp}(X)$, $y \in \mathrm{core}_d\, U \implies y \in U$.
(3) $\forall V \in \mathsf{ClopSUp}(X)$, $\max{\uparrow}y \subseteq V \implies y \in V$.
(4) $\{y\} = \max({\downarrow}x \cap Y)$ *for some* $x \in \max X$.

Proof. (1) ⇒ (2) If $y \in \mathrm{core}_d\, U$, then $y \in dU$. Therefore, since $y \in Y_d \subseteq N_d$, Lemma 3.10(1) implies that $y \in U$.

(2) ⇒ (3) Suppose $\max{\uparrow}y \subseteq V$. Then ${\uparrow}y \subseteq {\downarrow}V$. Since V is a clopen Scott upset, $V = \mathrm{core}\,V$ by Remark 2.14. Therefore, ${\downarrow}V = {\downarrow}\,\mathrm{core}\,V$, and hence $y \in \mathrm{core}_d\, V$ by Lemma 4.10(2). Thus, $y \in V$ by (2).

(3) ⇒ (4) Suppose that for every $x \in \max{\uparrow}y$ there is $y' \in {\downarrow}x \cap Y$ with $y < y' \leq x$. Then $y' \not\leq y$, so Lemma 4.11 implies that there is $V_x \in \mathsf{ClopSUp}(X)$ with $y' \in V_x$ and $y \notin V_x$. Therefore, $\max{\uparrow}y \subseteq \bigcup V_x$, and since $\max{\uparrow}y$ is closed (see Lemma 2.6(4)) and the open cover is directed, there is $V \in \mathsf{ClopSUp}(X)$ containing $\max{\uparrow}y$ and missing y, a contradiction.

(4) ⇒ (1) It is sufficient to show that $da \in y$ implies $a \in y$ for each $a \in L$, and hence it is enough to show that $y \in dU$ implies $y \in U$ for each $U \in \mathsf{ClopUp}(X)$. Let $y \in dU$. Then $y \in \mathrm{cl}\,\mathrm{core}_d\, U$ since d is inductive. Therefore, $y \in \mathrm{core}_d\, U$ by Lemma 4.5(2). Thus, by Lemma 4.10(1), $y \in dV = V^{**}$ for some $V \in \mathsf{ClopSUp}(X)$ with $V \subseteq U$. Hence, ${\uparrow}y \subseteq {\downarrow}V$. By (4), there is $x \in \max X$ with $\{y\} = \max({\downarrow}x \cap Y)$. But then $x \in V$, and since V is a Scott upset, there is $y' \in V \cap Y$ with $y' \leq x$. Consequently, $y' \leq y$, and so $y \in V \subseteq U$. □

5. $\max Y$ and Regularity of L_d

Martinez and Zenk [28, Proposition 5.2] characterized when L_d is a regular frame. In this section, we give several alternative characterizations, utilizing Priestley duality. This, in particular, involves the maximal spectrum $\max Y$ of the localic part Y of the Priestley space of L. As a consequence, we obtain that L_d is regular iff L_d is locally Stone.

Recall (see, e.g. [31, p. 89]) that a frame L is *regular* if for all $a \in L$ we have

$$a = \bigvee\{b \in L \mid b^* \vee a = 1\}.$$

Priestley spaces of regular frames were studied in [6, 10, 36]. We recall:

Definition 5.1 ([10, Definitions 7.1 and 7.6]). Let X be an L-space.

(1) For $U \in \mathsf{ClopUp}(X)$, the *regular part* of U is

$$\operatorname{reg} U = \bigcup \{V \in \mathsf{ClopUp}(X) \mid {\downarrow} V \subseteq U\}.$$

(2) X is *L-regular* if $\operatorname{cl} \operatorname{reg} U = U$ for each $U \in \mathsf{ClopUp}(X)$.

Theorem 5.2. *Let L be a frame, X its Priestley space, and Y the localic part of X.*

(1) [6, Lemma 3.6] *L is regular iff X is L-regular.*
(2) [10, Proof of Theorem 7.11] *If X is an SL-space and Y is regular, then X is L-regular.*
(3) [10, Lemma 7.15(3)] *If X is L-regular, then $Y \subseteq \min X$.*

An element $p \neq 1$ of a frame L is (*meet-*)*prime* if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$ (see, e.g. [31, p. 13]). A prime element p is *minimal prime with respect to $a \in L$* if p is minimal among the primes $q \geq a$. It is known (see, e.g. [30, p. 264]) that every prime element p greater than $a \in L$ has a minimal prime element q with respect to a beneath it. Since the assignment $p \mapsto L \backslash {\downarrow} p$ establishes an isomorphism between the posets of prime elements and completely prime filters (see, e.g. [31, p. 14]), this condition can equivalently be formulated as follows: for every completely prime filter P contained in a filter F, there exists a completely prime filter Q that is maximal among the completely prime filters contained in F. Thus, we arrive at the following lemma, which gives the means to find (relatively) maximal localic points.

Lemma 5.3. *Let L be a frame, X its Priestley space, Y the localic part of X, and $y \in Y$.*

(1) ${\uparrow} y \cap \max Y \neq \varnothing$.
(2) ${\uparrow} y \cap \max({\downarrow} x \cap Y) \neq \varnothing$ *for every $x \in X$ with $y \leq x$.*

Proof. For (1) take $P = y$ and $F = L$, and for (2) take $P = y$ and $F = x$. □

We show that $\max Y \subseteq Y_d$, but that the converse is not true in general. For this we require the following lemma.

Lemma 5.4. *Let L be an arithmetic frame, X its Priestley space, and Y the localic part of X.*

(1) *If F and G are Scott upsets of X, then so is $F \cap G$.*
(2) *Suppose $y \in \max Y$ and F is a Scott upset of X. If ${\uparrow} y \cap F \neq \varnothing$, then $y \in F$.*

Proof. (1) This can be seen by applying [11, Lemma 5.2; 10, Lemma 6.3(2)]. To keep the proof self-contained, we give a short argument. By Lemma 4.11, F and G are intersections of down-directed families of clopen Scott upsets. Therefore, so is $F \cap G$ since the binary intersection of clopen Scott upsets is a clopen Scott upset (X is an arithmetic L-space). Thus, $F \cap G$ is a Scott upset because the intersection of a down-directed family of clopen Scott upsets is a Scott upset (see [10, Lemma 5.14(1)]).

(2) Since $\uparrow y \cap F$ is a nonempty closed upset, $\min(\uparrow y \cap F) \neq \varnothing$ (see, e.g. [20, Theorem 3.2.1]). Because $\uparrow y, F$ are Scott upsets, $\uparrow y \cap F$ is also a Scott upset by (1). Therefore, $\min(\uparrow y \cap F) \subseteq Y$. Since y is underneath each point in $\min(\uparrow y \cap F)$ and $y \in \max Y$, we conclude that $\min(\uparrow y \cap F) = \{y\}$, and hence $y \in F$. □

Theorem 5.5. *Let X be an algebraic L-space and Y its localic part.*

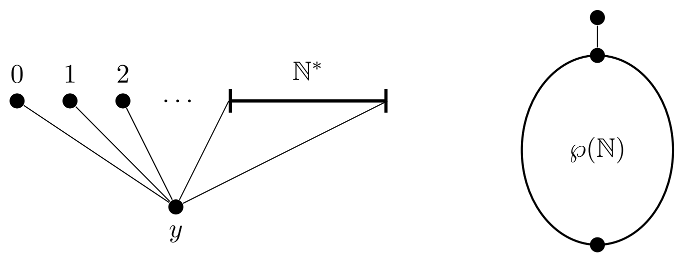
(1) *If $N \subseteq X$ is a cofinal inductive nuclear subset, then* $\max Y \subseteq N$.
(2) *If X is the Priestley space of an arithmetic frame L, then* $\max Y \subseteq Y_d$.

Proof. (1) Let $y \in \max Y$. Since $\uparrow y$ is a Scott upset and N is inductive, $\uparrow(\uparrow y \cap N)$ is a Scott upset. Because N is cofinal, $\max X \subseteq N$, and thus $\uparrow y \cap N \neq \varnothing$. Therefore, $\uparrow(\uparrow y \cap N) \subseteq \uparrow y$ is a nonempty Scott upset, so $y \in \uparrow(\uparrow y \cap N)$ by Lemma 5.4(2). Hence, $\uparrow y \subseteq \uparrow(\uparrow y \cap N)$. Consequently, $\uparrow y = \uparrow(\uparrow y \cap N)$, and so $y \in N$.

(2) By Proposition 4.9(2), N_d is a cofinal inductive nuclear subset of X. Therefore, $\max Y \subseteq N_d$ by (1), and so $\max Y \subseteq Y_d$ by Proposition 4.9(1). □

Example 5.6. To see that in general $\max Y \neq Y_d$, let $\beta\mathbb{N}$ be the Stone–Čech compactification of the natural numbers (see, e.g. [18, p. 174]). As is customary, we write $\mathbb{N}^*$ for the remainder. Let $X = \beta\mathbb{N} \cup \{y\}$, where the order on X is defined as shown in Fig. 1(a). It is well known (see, e.g. [17, p. 28]) that $\beta\mathbb{N}$ is homeomorphic to the Stone space of the powerset $\wp(\mathbb{N})$ of $\mathbb{N}$. Therefore, X is homeomorphic to the Priestley space of the lattice L obtained by adding a new top to $\wp(\mathbb{N})$; see Fig. 1(b). Since $\wp(\mathbb{N})$ is an arithmetic frame and $1 \in K(L)$, it is clear that so is L, and hence X is an arithmetic L-space.

Because the set of isolated points of X is $\mathbb{N} \cup \{y\}$, we have that $\downarrow x$ is clopen iff x is an isolated point of X. Thus, the localic part of X is $Y = \mathbb{N} \cup \{y\}$. Therefore, $\max Y = \mathbb{N}$. On the other hand, $y \in N_d$ by Theorem 4.12(4), so $Y_d = Y$. Consequently, $Y_d \neq \max Y$.



(a) The Priestley space $\beta\mathbb{N} \cup \{y\}$. (b) The lattice $\wp(\mathbb{N})$ with a new top.

Fig. 1. An arithmetic L-space in which $Y_d \neq \max Y$.

In order to characterize when L_d is regular, we require the following lemma, which allows us to separate Scott upsets from maximal localic points via clopen Scott upsets.

Lemma 5.7. *Let X be an arithmetic L-space, Y its localic part, and F a Scott upset of X. If $y \in \max Y \backslash F$, then there is $U \in \mathsf{ClopSUp}(X)$ with $y \in U$ and $U \cap F = \varnothing$.*

Proof. Since $y \notin F$, we have ${\uparrow} y \cap F = \varnothing$ by Lemma 5.4(2). Therefore, Lemma 4.11 yields that $\bigcap\{U \in \mathsf{ClopSUp}(X) \mid y \in U\} \cap F = \varnothing$. Thus, we can use compactness and the fact that finite intersections of clopen Scott upsets are again Scott upsets (X is an arithmetic L-space) to produce $U \in \mathsf{ClopSUp}(X)$ with $y \in U$ and $U \cap F = \varnothing$. □

We recall that a space X is *locally Stone* if X is zero-dimensional, locally compact, and Hausdorff. Thus, X is locally Stone if in the definition of a Stone space we weaken compactness to local compactness. A frame L is *locally Stone* if it is isomorphic to the frame of opens of a locally Stone space. By [8, Theorem 3.11], a frame is locally Stone iff it is algebraic and zero-dimensional (each element is a join of complemented elements). We will use the following fact: if L is an algebraic frame and X its Priestley space, then the localic part Y of X is locally compact (see [10, Theorem 5.10], where the result is proved in the more general setting of continuous frames).

Theorem 5.8. *Let L be an arithmetic frame, X its Priestley space, and Y the localic part of X. The following are equivalent.*

(1) *L_d is regular.*
(2) *N_d is L-regular.*
(3) *Y_d is an antichain.*
(4) $\max Y = Y_d$.
(5) *Y_d is a locally Stone space.*
(6) *L_d is a locally Stone frame.*

Proof. (1) ⇔ (2) This is immediate from Theorem 5.2(1) since N_d is the Priestley space of L_d.

(2) ⇒ (3) If N_d is L-regular, then $Y_d \subseteq \min N_d$ by Theorem 5.2(3). Therefore, Y_d is an antichain.

(3) ⇒ (4) By Theorem 5.5(2), $\max Y \subseteq Y_d$. For the converse, suppose $y \in Y_d$. By Lemma 5.3(1), there is $y' \in \max Y \cap {\uparrow} y$. Then $y' \in \max Y \subseteq Y_d$, so $y = y'$ since Y_d is an antichain by (3). Thus, $y \in \max Y$.

(4) ⇒ (5) Since L is arithmetic, L_d is arithmetic by Theorem 4.8(3). Therefore, as we pointed out above, Y_d is locally compact. Recall (see Remark 2.9) that open subsets of Y_d are exactly the sets of the form $U \cap Y_d$ for $U \in \mathsf{ClopUp}(X)$. Hence, (4) implies that open subsets are the sets of the form $U \cap \max Y$. By Lemma 4.5(2),

$$U \cap \max Y = \mathrm{cl}(\mathrm{core}\, U) \cap \max Y = \mathrm{core}\, U \cap \max Y.$$

Thus, to see that Y_d is zero-dimensional, it is enough to show that $U \cap \max Y$ is clopen for each $U \in \mathsf{ClopSUp}(X)$. For this it is sufficient to show that for each $y \in \max Y \backslash U$ there is $V \in \mathsf{ClopSUp}(X)$ with $y \in V \cap \max Y \subseteq \max Y \backslash U$. But this follows from Lemma 5.7. Finally, to see that Y_d is Hausdorff, let $y, y' \in Y_d = \max Y$ be distinct. Then $y \notin {\uparrow} y'$, so by Lemma 5.7 there is $U \in \mathsf{ClopSUp}(X)$ such that $y \in U$ and $y' \notin U$. But then $y \in U \cap Y_d$, which is clopen by the above.

(5) ⇔ (6) Because L_d is spatial and Y_d is the space of points of L_d (see Remark 2.9), L_d is isomorphic to the frame $\Omega(Y_d)$ of open subsets of Y_d. Therefore, Y_d is locally Stone iff L_d is locally Stone by [8, Theorem 3.11].

(5) ⇒ (2) Since Y_d is locally Stone, Y_d is regular. Hence, N_d is L-regular by Theorem 5.2(2). □

6. Spectra of Maximal *d*-Elements

In this section, we begin our investigation of the spectrum $\max L_d$ of maximal d-elements of an arithmetic frame L, as introduced in [13]. First, we show that $\max L_d$ is in a bijective correspondence with $\min Y_d$. Following this, we establish that $\min Y_d$, viewed as a subspace of Y, is homeomorphic to $\max L_d$. The homeomorphism enables us to analyze the properties of $\max L_d$ through $\min Y_d$. We show that the frame $\Omega(\min Y_d)$ of open subsets of $\min Y_d$ can be realized as a sublocale of L, and describe the corresponding nuclear subset of X. We conclude the section by observing that the localic part of this nuclear subset is the soberification of $\min Y_d$.

Definition 6.1. Let X be an arithmetic L-space and $U \in \mathsf{ClopUp}(X)$.

(1) We call U a *d-upset* if $\operatorname{cl}\operatorname{core}_d U = U$.
(2) We call U a *maximal d-upset* if it is maximal among proper d-upsets of X.

Remark 6.2. Since d is inductive, it is straightforward to verify that (maximal) d-elements of an arithmetic frame correspond to (maximal) d-upsets of its Priestley space.

We now show that maximal d-upsets are in one-to-one correspondence with elements of $\min Y_d$. For this, we require the following lemmas.

Lemma 6.3. *Let X be an arithmetic L-space and $U \in \mathsf{ClopUp}(X)$. Then $\operatorname{cl}\operatorname{core}_d U = X$ iff $Y_d \subseteq U$.*

Proof. Since d is inductive, $\operatorname{cl}\operatorname{core}_d U = X$ iff $dU = X$ (see Theorem 4.6(2)). It follows from Lemma 3.10(5) that $dU = X$ iff $N_d \subseteq U$. But by Proposition 4.9(4), $N_d = \operatorname{cl} Y_d$, so $N_d \subseteq U$ iff $Y_d \subseteq U$ since U is closed. □

Lemma 6.4. *Let X be an arithmetic L-space and $y \in Y_d$.*

(1) *$X \backslash {\downarrow} y$ is a d-upset.*
(2) *$X \backslash {\downarrow} y$ is a maximal d-upset iff $y \in \min Y_d$.*

(3) *Maximal d-upsets are exactly the clopen upsets of the form* $X\backslash{\downarrow}y$ *for some* $y \in \min Y_d$.

Proof. (1) We need to show $X\backslash{\downarrow}y = \mathrm{cl}\,\mathrm{core}_d(X\backslash{\downarrow}y)$. For this it suffices to show that $Y \cap (X\backslash{\downarrow}y) = Y \cap \mathrm{core}_d(X\backslash{\downarrow}y)$ since X is an SL-space. We have $Y \cap (X\backslash{\downarrow}y) \subseteq Y \cap \mathrm{core}_d(X\backslash{\downarrow}y)$ since $Y \cap U = Y \cap \mathrm{core}\,U$ and $\mathrm{core}\,U \subseteq \mathrm{core}_d\,U$ for each clopen upset U. For the reverse inclusion, let $z \in Y$ and suppose towards a contradiction that $z \in \mathrm{core}_d(X\backslash{\downarrow}y)$ and $z \notin X\backslash{\downarrow}y$. Then $z \le y$, so $y \in \mathrm{core}_d(X\backslash{\downarrow}y)$. Therefore, $y \in X\backslash{\downarrow}y$ by Theorem 4.12 ($y \in Y_d$), a contradiction.

(2) Suppose $y \notin \min Y_d$. Then there exists $y' \in Y_d$ with $y' < y$. Therefore, there exists $V \in \mathsf{ClopUp}(X)$ containing y and missing y'. Let $U = \mathrm{cl}\,\mathrm{core}_d(V \cup (X\backslash{\downarrow}y))$. Then U is a d-upset and $X\backslash{\downarrow}y \subsetneq U$ because $y \in U$ and $y \notin X\backslash{\downarrow}y$. But $U = \mathrm{cl}\,\mathrm{core}_d(V \cup (X\backslash{\downarrow}y)) \neq X$ by Lemma 6.3 since $y' \in Y_d$ and $y' \notin V \cup (X\backslash{\downarrow}y)$, yielding that $Y_d \not\subseteq V \cup (X\backslash{\downarrow}y)$. Thus, $X\backslash{\downarrow}y$ is not a maximal d-upset.

Suppose $y \in \min Y_d$ and $X\backslash{\downarrow}y \subsetneq U$ for a d-upset U. Then $y \in U$, and since $y \in \min Y_d$ we get $Y_d \subseteq (X\backslash{\downarrow}y) \cup \{y\} \subseteq U$. Therefore, $U = \mathrm{cl}\,\mathrm{core}_d\,U = X$ by Lemma 6.3.

(3) By (2) it suffices to show that every maximal d-upset is of the desired form, so suppose $U \in \mathsf{ClopUp}(X)$ is a maximal d-upset. Then $\mathrm{cl}\,\mathrm{core}_d\,U = U \neq X$, so $Y_d \not\subseteq U$ by Lemma 6.3. Therefore, there exists $y \in Y_d\backslash U$. Then ${\downarrow}y \cap U = \varnothing$, so $U \subseteq X\backslash{\downarrow}y \neq X$. But $X\backslash{\downarrow}y$ is a d-upset by (1). Hence, $U = X\backslash{\downarrow}y$ since U is a maximal d-upset. Thus, $y \in \min Y_d$ by (2). □

As an immediate consequence, we obtain:

Theorem 6.5. *Let* X *be an arithmetic L-space. The map* $y \mapsto X\backslash{\downarrow}y$ *is a bijection from* $\min Y_d$ *to the collection of maximal d-upsets of* X.

Equipping Y_d with the subspace topology inherited from Y, we have:

Theorem 6.6. $\min Y_d$ *is homeomorphic to* $\max L_d$.

Proof. Since $\varphi : L \to \mathsf{ClopUp}(X)$ is an isomorphism, define $\alpha : \min Y_d \to \max L_d$ by $\alpha(y) = \varphi^{-1}(X\backslash{\downarrow}y)$. By Theorem 6.5, α is a bijection. Thus, it suffices to show that for all $U \subseteq \min Y_d$ we have U is open iff $\alpha(U)$ is open. Now, U is open iff $U = V \cap \min Y_d$ for some $V \in \mathsf{ClopUp}(X)$, and $\alpha(U)$ is open iff

$$\alpha(U) = \{m \in \max L_d \mid a \not\le m\}$$

for some $a \in L$ (see, e.g. [13, Sec. 3]). Since $m \in \max(L_d)$ iff $\varphi(m)$ is a maximal d-upset, by Lemma 6.4(3) we have that $m \in \max(L_d)$ iff $\varphi(m) = X\backslash{\downarrow}y$ for some $y \in \min Y_d$. Moreover, for $a \in L$, we have that $\varphi(a) \not\subseteq X\backslash{\downarrow}y$ iff $y \in \varphi(a)$. Therefore,

$$\alpha(U) = \{m \in \max L_d \mid a \not\le m\} \Leftrightarrow \varphi[\alpha(U)] = \{\varphi(m) \mid \varphi(a) \not\subseteq \varphi(m)\}$$

$$\Leftrightarrow \varphi[\alpha(U)] = \{X\backslash{\downarrow}y \mid y \in \min Y_d,\ \varphi(a) \not\subseteq X\backslash{\downarrow}y\}$$

$$\Leftrightarrow U = \{y \in \min Y_d \mid y \in \varphi(a)\} \Leftrightarrow U = \varphi(a) \cap \min Y_d. \qquad \square$$

As we pointed out in the introduction, if L is an arithmetic frame with a unit (see the beginning of Sec. 7), then $\max L_d$ is a compact T_1-space. We next show that being T_1 does not depend on the existence of a unit.

Proposition 6.7. $\min Y_d$ *is* T_1.

Proof. Suppose $y, y' \in \min Y_d$ are distinct. Then $y \not\leq y'$ since $\min Y_d$ is an antichain. By Priestley separation, there exists $U \in \mathsf{ClopUp}(X)$ with $y \in U$ and $y' \notin U$. Hence, $U \cap \min Y_d$ is an open subset of $\min Y_d$ containing y and missing y'. □

We now concentrate on the frame $\Omega(\min Y_d)$ of open subsets of $\min Y_d$ and show that it can be realized as a sublocale of L. To this end, since L is isomorphic to $\mathsf{ClopUp}(X)$, we introduce a nucleus on $\mathsf{ClopUp}(X)$ that determines $\Omega(\min Y_d)$.

Lemma 6.8. *The map* $h : \mathsf{ClopUp}(X) \to \Omega(\min Y_d)$, *given by* $h(U) = U \cap \min Y_d$, *is an onto frame homomorphism.*

Proof. It is clear that h is onto and preserves finite meets. To see that it preserves arbitrary joins let $\{U_i\} \subseteq \mathsf{ClopUp}(X)$. Then, by Lemma 4.5(2),

$$h\left(\bigvee U_i\right) = \left(\mathrm{cl}\bigcup U_i\right) \cap \min Y_d = \left(\bigcup U_i\right) \cap \min Y_d$$
$$= \bigcup (U_i \cap \min Y_d) = \bigcup h(U_i).$$
□

By the previous lemma, there is a nucleus $\rho = h_* \circ h : \mathsf{ClopUp}(X) \to \mathsf{ClopUp}(X)$, where h_* is the right adjoint of h (see, e.g. [31, p. 31]). Then, for each $U \in \mathsf{ClopUp}(X)$,

$$\begin{aligned}\rho(U) &= \bigvee\{V \in \mathsf{ClopUp}(X) \mid h(V) \subseteq h(U)\}\\ &= \mathrm{cl}\bigcup\{V \in \mathsf{ClopUp}(X) \mid V \cap \min Y_d \subseteq U \cap \min Y_d\}\\ &= \mathrm{cl}\bigcup\{V \in \mathsf{ClopUp}(X) \mid V \cap \min Y_d \subseteq U\}.\end{aligned}$$

Lemma 6.9. *Let* L *be an arithmetic frame and* X *its* L*-space. For* $a \in L$ *set*

$$M_a = \bigwedge\{m \in \max L_d \mid a \leq m\}.$$

Then $\varphi(\bigwedge M_a) = \rho(\varphi(a))$.

Proof. Observe that

$$\begin{aligned}\varphi\left(\bigwedge M_a\right) &= \varphi\left(\bigvee\{b \in L \mid b \leq m \text{ for all } m \in M_a\}\right)\\ &= \mathrm{cl}\bigcup\left\{V \in \mathsf{ClopUp}(X) \mid V \subseteq \bigcap \varphi[M_a]\right\}.\end{aligned}$$

Recall that $m \in \max L_d$ iff $\varphi(m)$ is a maximal d-upset. Thus, using Lemma 6.4(3), we obtain that $m \in M_a$ iff $\varphi(m) = X \backslash {\downarrow} y$ for some $y \in \min Y_d \backslash \varphi(a)$.

Therefore,

$$\bigcap \varphi[M_a] = \bigcap \{X \backslash {\downarrow} y \mid y \in \min Y_d \backslash \varphi(a)\} = X \backslash {\downarrow}(\min Y_d \backslash \varphi(a)).$$

Consequently, since V is an upset,

$$\begin{aligned} V \subseteq \bigcap \varphi[M_a] &\Leftrightarrow V \subseteq X \backslash {\downarrow}(\min Y_d \backslash \varphi(a)) \\ &\Leftrightarrow V \cap {\downarrow}(\min Y_d \backslash \varphi(a)) = \varnothing \\ &\Leftrightarrow V \cap (\min Y_d \backslash \varphi(a)) = \varnothing \\ &\Leftrightarrow V \cap \min Y_d \subseteq \varphi(a), \end{aligned}$$

and the result follows from the above description of ρ. □

Remark 6.10. By the previous lemma, the nucleus ρ can be defined on an arbitrary arithmetic frame L by

$$\rho(a) = \bigwedge \{m \in \max L_d \mid a \leq m\}$$

for each $a \in L$.

We now describe the nuclear subset of X corresponding to the nucleus ρ. For this we use the following:

Theorem 6.11. *Let L be an arithmetic L-space. Then $N_\rho = \operatorname{cl} \min Y_d$.*

Proof. By Remark 3.7(4), $\operatorname{cl} \min Y_d$ is a nuclear subset of X. Let $j \in N(L)$ be the nucleus associated with $\operatorname{cl} \min Y_d$ (see Remark 3.5). It suffices to show that $\varphi(j(a)) = \rho(\varphi(a))$ for all $a \in L$.

($\subseteq$) Let $x \in \varphi(j(a))$. Then ${\uparrow} x \cap \operatorname{cl} \min Y_d \subseteq \varphi(a)$ by Lemma 3.10(2). Since ${\uparrow} x$ is a closed upset, it is an intersection of clopen upsets (see Lemma 2.6(2)). Therefore, by compactness, there is $V \in \mathsf{ClopUp}(X)$ such that $x \in V$ and $V \cap \operatorname{cl} \min Y_d \subseteq \varphi(a)$. Thus, $V \cap \min Y_d \subseteq \varphi(a)$, and so $x \in \rho(\varphi(a))$.

($\supseteq$) By Lemma 4.5(2),

$$\begin{aligned} \rho(\varphi(a)) \cap \min Y_d &= \operatorname{cl} \bigcup \{V \in \mathsf{ClopUp}(X) \mid V \cap \min Y_d \subseteq \varphi(a)\} \cap \min Y_d \\ &= \bigcup \{V \in \mathsf{ClopUp}(X) \mid V \cap \min Y_d \subseteq \varphi(a)\} \cap \min Y_d \subseteq \varphi(a). \end{aligned}$$

Therefore, since $\varphi(a)$ is closed,

$$\rho(\varphi(a)) \cap \operatorname{cl} \min Y_d \subseteq \operatorname{cl}(\rho(\varphi(a)) \cap \min Y_d) \subseteq \varphi(a).$$

Thus, for each $x \in \rho(\varphi(a))$,

$${\uparrow} x \cap \operatorname{cl} \min Y_d \subseteq \rho(\varphi(a)) \cap \operatorname{cl} \min Y_d \subseteq \varphi(a).$$

Consequently, $x \in \varphi(ja)$ by Lemma 3.10(2). □

We conclude this section by describing the link between $\min Y_d$ and the localic part of N_ρ.

Proposition 6.12. *The localic part Y_ρ of N_ρ is the soberification of* $\min Y_d$.

Proof. The localic part of any L-space is the space of points of the associated frame (see Remark 2.9). Thus, Y_ρ is the space of points of $\Omega(\min Y_d)$, which is the soberification of $\min Y_d$ (see, e.g. [25, p. 44]). □

7. Compactness of the Maximal *d*-Spectrum

We now turn our attention to studying topological properties of $\min Y_d$. As we mentioned in the introduction, in [13] the second author only considered arithmetic frames with a *unit*; that is, a compact dense element, where we recall (see the paragraph before Proposition 3.14) that an element $a \in L$ is *dense* if $a^{**} = 1$. In this section, we characterize units in the language of Priestley spaces and compare the existence of a unit to compactness of $\min Y_d$.

We start by characterizing compact subsets of $\min Y_d$ in terms of special Scott upsets of X.

Definition 7.1. Let X be an arithmetic L-space and Y its localic part. A subset $Z \subseteq X$ is called *d-initial* if $Z \cap Y \subseteq {\uparrow}(Z \cap \min Y_d)$.

Lemma 7.2. *Let X be an arithmetic L-space and Y its localic part.*

(1) *A Scott upset $F \subseteq X$ is d-initial iff $F = {\uparrow}(F \cap \min Y_d)$.*
(2) *If $F \subseteq X$ is a d-initial Scott upset, then $F \cap \min Y_d$ is compact.*
(3) *If $K \subseteq \min Y_d$ is compact then ${\uparrow}K$ is a d-initial Scott upset.*

Proof. (1) The right-to-left implication is immediate. For the left-to-right implication, let F be a d-initial Scott upset. Then $\min F \subseteq F \cap Y \subseteq {\uparrow}(F \cap \min Y_d)$, and hence $F = {\uparrow}\min F = {\uparrow}(F \cap \min Y_d)$.

(2) Suppose that $F \cap \min Y_d \subseteq \bigcup (U_i \cap \min Y_d)$ for a family $\{U_i\} \subseteq \mathsf{ClopUp}(X)$. Then $F \cap \min Y_d \subseteq \bigcup U_i$, and so $F \subseteq \bigcup U_i$ by (1). Since F is closed, it is compact, and hence $F \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$ for some $i_1, \ldots, i_n$. Therefore,

$$F \cap \min Y_d \subseteq (U_{i_1} \cap \min Y_d) \cup \cdots \cup (U_{i_n} \cap \min Y_d),$$

and hence $F \cap \min Y_d$ is compact.

(3) Clearly, ${\uparrow}K$ is d-initial. Thus, it suffices to show that ${\uparrow}K$ is a Scott upset. Since $\min {\uparrow}K = K \subseteq \min Y_d$, it is enough to show that ${\uparrow}K$ is closed. Let $x \notin {\uparrow}K$. Then $y \not\leq x$ for all $y \in K$. By Priestley separation, there is $U_y \in \mathsf{ClopUp}(X)$ such that $y \in U_y$ and $x \notin U_y$. Therefore, $K \subseteq \bigcup (U_y \cap \min Y_d)$, so by compactness of K there is $U \in \mathsf{ClopUp}(X)$ such that ${\uparrow}K \subseteq U$ and $x \notin U$. Thus, ${\uparrow}K$ is closed. □

As an immediate consequence, we have:

Proposition 7.3. *Let X be an arithmetic L-space and $K \subseteq \min Y_d$. The following are equivalent*:

(1) *K is compact.*
(2) *$\uparrow K$ is a d-initial Scott upset.*
(3) *There is a d-initial Scott upset $F \subseteq X$ such that $K = F \cap \min Y_d$.*

Theorem 7.4. *Let X be an arithmetic L-space. There is a poset isomorphism between compact subsets of $\min Y_d$ and d-initial Scott upsets of X (both ordered by inclusion).*

Proof. Consider the maps $F \mapsto F \cap \min Y_d$ and $K \mapsto \uparrow K$, where $F \subseteq X$ is a d-initial Scott upset and $K \subseteq \min Y_d$ is compact. These are well defined by Lemma 7.2, and are clearly order preserving. It suffices to show that these maps are inverses of each other. But $F = \uparrow(F \cap \min Y_d)$ by Lemma 7.2(1), and it is easy to see that $K = \uparrow K \cap \min Y_d$, completing the proof. □

We recall (see, e.g. [31, p. 25]) that a frame is *max-bounded* if each proper element is below a maximal element.

Proposition 7.5. *Let L be an arithmetic frame and X its Priestley space. Then L_d is max-bounded iff N_d is d-initial.*

Proof. Since L_d is max-bounded iff every proper d-upset is contained in a maximal d-upset (see Remark 6.2), it suffices to show that the latter condition is equivalent to N_d being d-initial.

($\Rightarrow$) Let $y \in N_d \cap Y$. Then $y \in Y_d$ by Proposition 4.9(1), so $X \backslash {\downarrow} y$ is a d-upset by Lemma 6.4(1). Hence, there exists a maximal d-upset U such that $X \backslash {\downarrow} y \subseteq U$. By Lemma 6.4(3), $U = X \backslash {\downarrow} y'$ for some $y' \in \min Y_d$. Thus, $X \backslash {\downarrow} y \subseteq X \backslash {\downarrow} y'$, which implies that $y' \in {\downarrow} y' \subseteq {\downarrow} y$. Therefore, $y' \leq y$, as required.

($\Leftarrow$) Let U be a proper d-upset. Then $U = \operatorname{cl} \operatorname{core}_d U \neq X$, so $Y_d \not\subseteq U$ by Lemma 6.3. Hence, there is $y \in Y_d \backslash U \subseteq N_d$. Since N_d is d-initial, there is $y' \in \min Y_d$ with $y' \leq y$. Therefore, $U \subseteq X \backslash {\downarrow} y'$, which is a maximal d-upset by Lemma 6.4(2). □

It is well known that if an arithmetic frame L has a unit, then L_d is max-bounded (see, e.g. [13, before Proposition 3.3]). The following example shows that L_d being max-bounded is a strictly weaker condition.

Example 7.6. Let $L = \wp(\mathbb{N})$. Then L is an arithmetic frame, and $da = a$ for all $a \in L$ since L is Boolean, so $L = L_d$. The maximal elements of L_d are exactly the coatoms. Therefore, L_d is max-bounded since it is atomic. However, L_d does not contain a unit since the only dense element is 1, which is not compact.

We end this section by describing the Priestley analogue of having a unit and its relation to compactness of $\min Y_d$.

Theorem 7.7. *Let L be an arithmetic frame and X its L-space. The following are equivalent.*

(1) *There is a unit in L.*
(2) *There is a cofinal $U \in \mathsf{ClopSUp}(X)$.*
(3) *There is $U \in \mathsf{ClopSUp}(X)$ such that $Y_d \subseteq U$.*

The previous conditions imply the following equivalent conditions.

(4) *$\uparrow \min Y_d$ is a Scott upset.*
(5) *$\min Y_d$ is compact.*

If in addition N_d is d-initial, then all five conditions are equivalent.

Proof. (1) $\Leftrightarrow$ (2) By Remark 2.14, $a \in K(L)$ iff $\varphi(a)$ is a Scott upset. Moreover, a is dense iff $\max X \subseteq \varphi(a)$ by Lemma 3.10(5) and Proposition 3.14(2).

(2) $\Leftrightarrow$ (3) Suppose $U \in \mathsf{ClopSUp}(X)$ is cofinal and $y \in Y_d$. Then

$$\max \uparrow y \subseteq \max X \subseteq U,$$

so $y \in U$ by Theorem 4.12(3). Therefore, $Y_d \subseteq U$. Conversely, suppose $Y_d \subseteq U$. Then $N_d = \operatorname{cl} Y_d \subseteq U$ since N_d is an SL-space. But N_d is cofinal by Lemma 3.16, so $\max X \subseteq U$.

(3) $\Rightarrow$ (4) Since N_d is inductive, $\uparrow(U \cap N_d)$ is a Scott upset. Therefore, $\min \uparrow(U \cap N_d) \subseteq Y$, so

$$\min \uparrow(U \cap N_d) \subseteq Y \cap N_d = Y_d.$$

We show that $\min \uparrow(U \cap N_d) = \min Y_d$. If $y \in \min \uparrow(U \cap N_d)$ and $y' \in Y_d$ is such that $y' \leq y$, then $y' \in U \cap N_d$ because $Y_d \subseteq U$, so $y = y'$ since

$$y \in \min \uparrow(U \cap N_d) = \min(U \cap N_d).$$

Hence, $\min \uparrow(U \cap N_d) \subseteq \min Y_d$. Conversely, if $y \in \min Y_d$ and $x \in \uparrow(U \cap N_d)$ with $x \leq y$, then there is $y' \in \min \uparrow(U \cap N_d) \subseteq Y_d$ with $y' \leq x \leq y$, so $y = y'$ since $y \in \min Y_d$. Consequently, $\min Y_d = \min \uparrow(U \cap N_d)$. Thus, $\uparrow \min Y_d = \uparrow(U \cap N_d)$, and hence $\uparrow \min Y_d$ is a Scott upset.

(4) $\Leftrightarrow$ (5) Since $\uparrow \min Y_d$ is d-initial, this follows from Proposition 7.3.

Finally, suppose that N_d is d-initial.

(5) $\Rightarrow$ (3) Since $\min Y_d$ is compact, the open cover $\{V \cap \min Y_d \mid V \in \mathsf{ClopSUp}(X)\}$ of $\min Y_d$ has a finite subcover, and since finite unions of clopen Scott upsets are clopen Scott upsets, there is $V \in \mathsf{ClopSUp}(X)$ such that $\min Y_d \subseteq V$. By Proposition 4.9(1), $Y_d = N_d \cap Y$. Thus, since N_d is d-initial, $Y_d \subseteq \uparrow \min Y_d \subseteq V$. $\square$

Remark 7.8. In Example 8.11, we will show that the assumption in Theorem 7.7 that N_d is d-initial is necessary. In fact, we will see that $\min Y_d$ may be compact Hausdorff without L having a unit.

8. Hausdorffness of the Maximal d-Spectrum

In this final section, we give an example of an arithmetic frame L with a unit such that $\max L_d$ is not Hausdorff, thus answering the question of [13] in the negative. In addition, we characterize exactly when $\max L_d$ is Hausdorff. Our characterization doesn't require that L has a unit, only that $\max L_d$ is locally compact. Under this assumption, we show that $\max L_d$ is Hausdorff iff it is stably locally compact, a condition that plays an important role in domain theory (see [21, Sec. VI.6]).

Example 8.1. Consider the Stone-Čech compactification of the natural numbers $\beta\mathbb{N}$. Partition the natural numbers in countably many countable subsets $\mathbb{N} = X_0 \cup X_1 \cup X_2 \cup \cdots$, where $X_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots\}$. Then for each X_i, $\operatorname{cl} X_i$ is a clopen set of $\beta\mathbb{N}$ homeomorphic to $\beta\mathbb{N}$ (see, e.g. [18, p. 174]). Let $X_i^* = \operatorname{cl}(X_i) \cap \mathbb{N}^*$ and let $Y_\omega = \{y_0, y_1, y_2, \ldots\} \cup \{\omega\}$ be the one-point compactification of a copy of the natural numbers. Consider now the disjoint union $X = \beta\mathbb{N} \sqcup Y_\omega$ and the partial order in Fig. 2, where $X_\omega^* = \mathbb{N}^* \setminus \bigcup_{n\in\mathbb{N}} X_n^*$.

Our goal is to show that X is an arithmetic L-space such that $\min Y_d$ is not Hausdorff. We have several things to verify.

Claim 8.2. *X is a Priestley space.*

Proof. X is a Stone space since it is the disjoint union of two Stone spaces. It remains to be shown that X satisfies the Priestley separation axiom. For $x \in \beta\mathbb{N}$ and $x' \in X$ with $x \not\leq x'$, finding a clopen upset containing x and missing x' is easy since $\beta\mathbb{N}$ is a clopen upset of X.

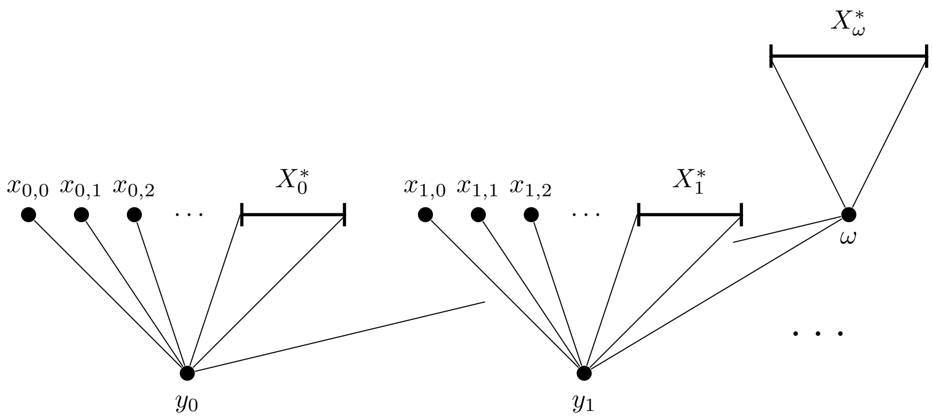


Fig. 2. The Priestley space of an arithmetic frame whose maximal d-spectrum is not Hausdorff.

Let $y \in Y_\omega$ and $x \in X$ with $y \not\leq x$. Then $x \notin {\uparrow}\omega$, so $x \in {\downarrow}(\mathrm{cl}\, X_i)$ for some i such that $y \neq y_i$. Consider $U = X \setminus ({\downarrow}\, \mathrm{cl}\, X_i)$. Since $\mathrm{cl}\, X_i$ is clopen, so is ${\downarrow}\, \mathrm{cl}\, X_i$. Thus, U is a clopen upset separating y from x. □

Claim 8.3. *X is an L-space.*

Proof. It is sufficient to show that $\mathrm{cl}\, U$ is a clopen upset for each open upset $U \subseteq X$. Let U be an open upset. Then $U \cap Y_\omega$ is open, and it is either empty or it must contain ω. In both cases, $\mathrm{cl}(U) \cap Y_\omega = U \cap Y_\omega$, so $\mathrm{cl}\, U = U \cup \mathrm{cl}(U \cap \beta\mathbb{N})$, which is clearly a clopen upset. □

Claim 8.4. *The localic part of X is $Y = Y_\omega \cup \mathbb{N}$.*

Proof. We have $Y \cap \mathbb{N}^* = \varnothing$ since ${\downarrow}x \cap \beta\mathbb{N} = \{x\}$ is not open for all $x \in \mathbb{N}^*$, so $Y \subseteq Y_\omega \cup \mathbb{N}$. For the converse, if $y \in Y_\omega \setminus \{\omega\}$ then ${\downarrow}y = \{y\}$ is clopen. Also, ${\downarrow}\omega = Y_\omega$ is clopen, so $Y_\omega \subseteq Y$. For $x \in X_i$, we have ${\downarrow}x = \{x, y_i\}$ is clopen. □

Claim 8.5. *Let $U \subseteq X$ be an upset. Then $U \in \mathsf{ClopSUp}(X)$ iff one of the following two conditions holds.*

(1) *U is a finite subset of $\mathbb{N}$.*
(2) *$U \cap Y_\omega$ is cofinite, and $y_i \notin U$ implies $\mathrm{cl}(X_i) \cap U$ is a finite subset of X_i.*

Proof. ($\Rightarrow$) Suppose $U \in \mathsf{ClopSUp}(X)$ and $U \cap Y_\omega$ is not cofinite. Since $U \cap Y_\omega$ is clopen and not cofinite, $\omega \notin U$. Hence, $U \cap Y_\omega = \varnothing$ since it is an upset. Therefore, $U \subseteq \beta\mathbb{N} = \mathrm{cl}\, \mathbb{N}$. By Lemma 4.5(1), $U \subseteq \mathbb{N}$ and by compactness U is finite. Suppose now that $U \cap Y_\omega$ is cofinite. If $y_i \notin U$, then $X_i^* \cap U = \varnothing$, since otherwise U can't be Scott upset because $\min U \not\subseteq Y$ (see Claim 8.3). Therefore, $U \cap \mathrm{cl}\, X_i \subseteq X_i$, and it has to be finite since it is compact.

($\Leftarrow$) If U is a finite subset of $\mathbb{N}$, then $U \in \mathsf{ClopUp}(X)$ and $U \subseteq Y$ by Claim 8.4, so $U \in \mathsf{ClopSUp}(X)$. Now suppose (2) holds. Then $\omega \in U$ and $y_i \notin U$ implies $X_i^* \cap U = \varnothing$. Hence, $\min U \subseteq Y_\omega \cup \mathbb{N} = Y$, so it suffices to show that U is clopen. Since $U \cap Y_\omega$ is cofinite it is clopen in Y_ω. Moreover,

$$U \cap \beta\mathbb{N} = \bigcup\{\mathrm{cl}(X_i) \cap U \mid y_i \notin U\} \cup \bigcup\{\mathrm{cl}\, X_i \mid y_i \in U\} \cup X_\omega^*.$$

By (2), $\bigcup\{\mathrm{cl}(X_i) \cap U \mid y_i \notin U\} = \bigcup\{X_i \cap U \mid y_i \notin U\} \subseteq \mathbb{N}$ is finite and hence clopen. Also, $\beta\mathbb{N} \setminus (\bigcup\{\mathrm{cl}\, X_i \mid y_i \in U\} \cup X_\omega^*)$ is clopen since only finitely many $y_i \notin U$. Therefore, $\bigcup\{\mathrm{cl}\, X_i \mid y_i \in U\} \cup X_\omega^*$ is clopen. Thus, $U \cap \beta\mathbb{N}$ is clopen, and so $U = (U \cap Y_\omega) \cup (U \cap \beta\mathbb{N})$ is clopen. □

Claim 8.6. *X is an algebraic L-space.*

Proof. Suppose $U \in \mathsf{ClopUp}(X)$. We need to show $U \subseteq \mathrm{cl}\, \mathrm{core}\, U$, so suppose $x \in U$. We consider three cases.

(i) If $x \in Y_\omega$ then $\omega \in \uparrow x \subseteq U$, so $U \cap Y_\omega$ is cofinite. Then

$$x \in \uparrow(U \cap Y_\omega) \in \mathsf{ClopSUp}(X)$$

by Claim 8.5(2), and $\uparrow(U \cap Y_\omega) \subseteq \operatorname{core} U$.

(ii) If $x \in \mathbb{N}$, then $\uparrow x = \{x\} \in \mathsf{ClopSUp}(X)$ by Claim 8.5(1), and $\uparrow x \subseteq \operatorname{core} U$.

(iii) Suppose $x \in \mathbb{N}^*$. Since U is clopen in X, $U \cap \beta\mathbb{N}$ is clopen, and therefore

$$\mathrm{cl}(U \cap \mathbb{N}) = U \cap \beta\mathbb{N}.$$

Hence, $x \in \mathrm{cl}(U \cap \mathbb{N})$. Suppose now that V is a clopen neighborhood of x. Then $V \cap (U \cap \mathbb{N}) \neq \varnothing$, but $U \cap \mathbb{N} = \bigcup\{\{n\} \mid n \in \mathbb{N} \cap U\} \subseteq \operatorname{core} U$ since $\{n\} \in \mathsf{ClopSUp}(X)$ by Claim 8.5(1). Therefore, $x \in \mathrm{cl}\operatorname{core} U$. □

Claim 8.7. *X is an arithmetic L-space.*

Proof. It suffices to show that $U \cap V \in \mathsf{ClopSUp}(X)$ for $U, V \in \mathsf{ClopSUp}(X)$ (see Definition 2.15(1)), so suppose $U, V \in \mathsf{ClopSUp}(X)$. Then U and V satisfy one of the two conditions of Claim 8.5. If either U or V is a finite subset of $\mathbb{N}$, then so is their intersection. Suppose U and V both satisfy Claim 8.5(2). Since a finite intersection of cofinite sets is cofinite, $U \cap V \cap Y_\omega$ is cofinite. If $y_i \notin U \cap V$, then either $y_i \notin U$ or $y_i \notin V$. Without loss of generality we may assume the former. Then $\mathrm{cl}(X_i) \cap U \cap V \subseteq \mathrm{cl}(X_i) \cap U$ is a finite subset of X_i. Thus, Claim 8.5(2) holds for $U \cap V$, and so $U \cap V \in \mathsf{ClopSUp}(X)$. □

Claim 8.8. $\min Y_d = Y_\omega \backslash \{\omega\}$.

Proof. Observe that for each $y \in Y = \mathbb{N} \cup Y_\omega$, there is $x \in \max X$ with $\{y\} = \max(\downarrow x \cap Y)$. Therefore, $Y_d = \mathbb{N} \cup Y_\omega$ by Theorem 4.12(4). Consequently, $\min Y_d = Y_\omega \backslash \{\omega\}$. □

Claim 8.9. $\min Y_d$ *is not Hausdorff.*

Proof. $\mathsf{ClopSUp}(X)$ forms a basis of $\min Y_d$ because $U \cap Y = \operatorname{core} U \cap Y$ for each $U \in \mathsf{ClopUp}(X)$. Consequently, it follows from Claim 8.5(2) that $\min Y_d$ is equipped with the cofinite topology, which is not Hausdorff since $\min Y_d$ is infinite. □

Claims 8.7 and 8.9 yield the following:

Theorem 8.10. *There exist arithmetic L-spaces X such that* $\min Y_d$ *is not Hausdorff.*

As promised in Remark 7.8, we now demonstrate that $\min Y_d$ can be compact Hausdorff without L having a unit.

Example 8.11. Redefine the order in the space X of Example 8.1 as in Fig 3. A similar reasoning to the above yields that X is an arithmetic L-space and its

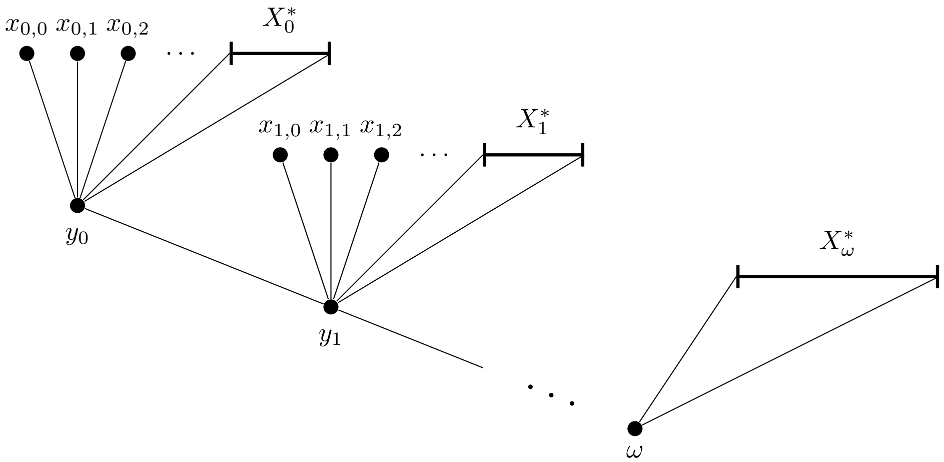


Fig. 3. The Priestley space of an arithmetic frame without a unit whose maximal d-spectrum is compact Hausdorff.

localic part is $Y := \mathbb{N} \cup (Y_\omega \backslash \{\omega\})$. Observe that $Y_d = Y$ and $\min Y_d = \varnothing$, so it is trivially compact Hausdorff. However, X has no cofinal clopen Scott upset since $X_\omega^* \subseteq \max X$ and ${\downarrow} X_\omega^* \cap Y = \varnothing$. Consequently, X is the L-space of an arithmetic frame without units.

To characterize when $\min Y_d$ is Hausdorff, we recall (see, e.g. [21, Definition VI-6.7]) that a topological space X is *coherent* if the binary intersection of compact saturated sets is compact. The space X is *stably locally compact* if it is sober, locally compact, and coherent. A stably locally compact space is *stably compact* if it is compact, and *spectral* if in addition compact open sets form a basis. It is well known (see, e.g. [25, p. 75]) that a spectral space is Hausdorff iff it is T_1. The next lemma generalizes this result to stably locally compact spaces.

Lemma 8.12. *If X is stably locally compact, then X is Hausdorff iff X is T_1.*

Proof. We only need to show the right-to-left implication. Suppose $x, y \in X$ are distinct. Let $\mathcal{K}_x = \{K \subseteq X \mid K \text{ is a compact saturated neighborhood of } x\}$ and define $\mathcal{K}_y$ similarly. It suffices to show that there exist $K_x \in \mathcal{K}_x$ and $K_y \in \mathcal{K}_y$ such that $K_x \cap K_y = \varnothing$. Since X is T_1 and locally compact, for each $z \in X$ distinct from x, there is a compact saturated neighborhood K of x missing z. Therefore, $\bigcap \mathcal{K}_x = \{x\}$, and similarly $\bigcap \mathcal{K}_y = \{y\}$. Consequently, $\bigcap \mathcal{K}_x \cap \bigcap \mathcal{K}_y = \varnothing$. By [21, Lemma VI-6.4], there exists a finite $\mathcal{K} \subseteq \mathcal{K}_x \cup \mathcal{K}_y$ such that $\bigcap \mathcal{K} = \varnothing$. Since X is stably locally compact, $\mathcal{K}_x$ and $\mathcal{K}_y$ are directed. Therefore, there are $K_x \in \mathcal{K}_x$ and $K_y \in \mathcal{K}_y$ such that $K_x \cap K_y = \varnothing$. Thus, X is Hausdorff. $\square$

Theorem 8.13. *Let X be an arithmetic L-space such that $\min Y_d$ is locally compact. Then $\min Y_d$ is Hausdorff iff $\min Y_d$ is stably locally compact.*

Proof. First suppose that X is Hausdorff. Then X is sober (see, e.g. [25, p. 43]). Let $K, M \subseteq X$ be compact saturated. Since X is Hausdorff, K, M are closed, so $K \cap M$ is closed. Since it is a closed subset of a compact set, $K \cap M$ must be compact. Thus, X is stably locally compact. Conversely, since $\min Y_d$ is T_1 by Proposition 6.7, $\min Y_d$ is Hausdorff by Lemma 8.12. □

Corollary 8.14. *Let L be an arithmetic frame with a unit and X its L-space.*

(1) $\min Y_d$ *is Hausdorff iff* $\min Y_d$ *is stably locally compact.*
(2) $\max L_d$ *is Hausdorff iff* $\max L_d$ *is stably locally compact.*

Proof. We only prove (1) as (2) follows from (1) and Theorem 6.6. Since L has a unit, $\min Y_d$ is compact by Theorem 7.7. First suppose that $\min Y_d$ is Hausdorff. Then $\min Y_d$ is compact Hausdorff, and hence $\min Y_d$ is stably locally compact. Conversely, if $\min Y_d$ is stably locally compact, then Theorem 8.13 applies, and hence $\min Y_d$ is Hausdorff. □

We conclude the paper with several interesting open problems:

- It remains open whether Theorem 8.13 can be reformulated as an equivalence between sobriety and Hausdorffness. Note that in Example 8.1, $\min Y_d$ is locally compact and coherent, but it fails to be Hausdorff solely because it is not sober.
- It also remains open whether $\min Y_d$ is always locally compact (and/or coherent). In the absence of sobriety, local compactness of $\min Y_d$ is not equivalent to $\Omega(\min Y_d)$ being a continuous frame (see, e.g. [25, p. 310]). This disparity emphasizes the importance of sobriety in these considerations. Indeed, it is plausible that in this setting sobriety alone implies Hausdorffness.
- Resolving the above questions requires developing a general method for identifying which topological spaces can be realized as $\min Y_d$. While each Stone space can be realized as such, it remains open whether the same can be said about each compact Hausdorff space (we note that it follows from [22] that each compact Hausdorff quasi F-space can be realized this way).

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