Algebraic Frames in Priestley duality

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Algebraic lattices

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- [Nachbin, 1949] Algebraic lattices are exactly the ideal lattices of join-semilattices.

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Frames are the focus of study in pointfree topology, as they generalize lattices of open sets of topological spaces.

There are several important examples of algebraic frames:

- Arithmetic frames also known as M-frames
  (compact elements form a sublattice)
- Coherent frames
  (compact elements form a bounded sublattice)
- Stone frames
  (compact elements form a boolean subalgebra)

Adjunction

There is a well-known dual adjunction between the following categories:

- **Top** topological spaces and continuous maps
- **Frm** frames and frame homomorphisms

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There is a well-known dual adjunction between the following categories:

- Top — topological spaces and continuous maps
- Frm — frames and frame homomorphisms

This adjunction restricts to a dual equivalence between the following subcategories. (Recall, a frame L is spatial if it is isomorphic to the frame of opens of a space.)

- Sob — full subcategory of Top consisting of sober spaces
- SFrm — full subcategory of Frm consisting of spatial frames

Theorem (Dowker-Papert, 1966)

Sob *and* SFrm *are dually equivalent.*

Compactly based spaces

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KBSob compactly based sober spaces and coherent maps

AlgFrm algebraic frames and coherent frame homomorphisms

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Now we have the following categories.

KBSob	compactly based sober spaces and coherent maps
AlgFrm	algebraic frames and coherent frame homomorphisms

Theorem (Hofmann-Keimel, 1972)

KBSob *and* AlgFrm *are dually equivalent.*

Dualities

Restricting to the full subcategories of arithmetic frames, coherent frames, and Stone frames yields the following dualities:

$$
\begin{array}{ccccccc}
\mathsf{AlgFrm} & \geqslant & \mathsf{AriFrm} & \geqslant & \mathsf{CohFrm} & \geqslant & \mathsf{StoneFrm} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathsf{KBSob} & \geqslant & \mathsf{SKBSp} & \geqslant & \mathsf{Spec} & \geqslant & \mathsf{Stone}
\end{array}
$$

where we have the following full subcategories of KBSob:

- **SKBSp** stably compactly based spaces
  (intersection of two compact opens is compact)
- **Spec** spectral spaces
  (stably compactly based + compact)
- **Stone** Stone spaces
  (spectral + zero-dimensional)

Priestley duality

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- **Pries** Priestley spaces and order-preserving continuous maps

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Consider the following categories:

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Let LPries be the category of L-spaces and L-morphisms.

Theorem (Pultr-Sichler, 1988)

LPries *and* Frm *are dually equivalent.*

Spatiality

Let L be a frame and X the Priestley space of L.

Definition

1. The spatial part of X is $Y := \{x \in X \mid {\downarrow}x \text{ is clopen}\}$.
2. X is an SL-space if Y is dense in X.
3. $\mathsf{SLPries}$ is the full subcategory of LPries of SL-spaces.

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Theorem

1. *L is spatial exactly when Y is dense in X.*
2. SLPries *is equivalent to* Sob *and dually equivalent to* SFrm.

Scott upsets

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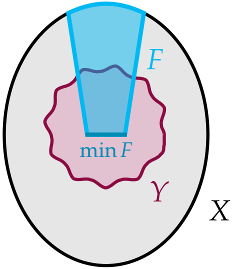

Algebraic L-spaces

For $U \in \mathsf{ClopUp}(X)$, let $\operatorname{core} U = \bigcup\{V \in \mathsf{ClopSUp}(X) \mid V \subseteq U\}$.

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An L-space is algebraic if $\operatorname{core} U$ is dense in U for each clopen upset U.

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An L-space is algebraic if $\operatorname{core} U$ is dense in U for each clopen upset U.

Theorem

Let L be a frame, X its Priestley space, and Y the spatial part of X. The following are equivalent.

1. *L is an algebraic frame.*
2. *X is an algebraic L-space.*
3. *Y is a compactly based sober space.*

Algebraic L-spaces

Definition

An L-morphism $f : X_1 \to X_2$ is coherent if

$$f^{-1}(\mathrm{core}\, U) \subseteq \mathrm{core}\, f^{-1}(U) \quad \text{for all } U \in \mathsf{ClopUp}(X_2).$$

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Let AlgLPries be the category of algebraic L-spaces and coherent L-morphisms.

Theorem

AlgLPries *is equivalent to* KBSob *and dually equivalent to* AlgFrm.

Arithmetic L-spaces

Definition

An algebraic L-space is arithmetic if

$$\operatorname{core}(U \cap V) = \operatorname{core} U \cap \operatorname{core} V \quad \text{for all } U, V \in \mathsf{ClopUp}(X).$$

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Let AriLPries be the full subcategory of AlgLPries consisting of arithmetic L-spaces.

Theorem

AriLPries *is equivalent to* SKBSp *and dually equivalent to* AriFrm.

Coherent L-spaces

Definition

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2. An arithmetic L-space is coherent if it is L-compact.

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Let CohLPries be the full subcategory of AriLPries consisting of coherent L-spaces.

Theorem

CohLPries *is equivalent to* Spec *and dually equivalent to* CohFrm.

Zero-dimensional L-spaces

Recall, a frame is zero-dimensional if every element is the join of complemented elements below it. These elements correspond to clopen upsets that are also downsets; we call such sets clopen bisets.

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1. $\operatorname{cen} U := \bigcup\{V \in \mathsf{ClopBi}(X) \mid V \subseteq U\}$.
2. X is a zero-dimensional L-space if $\operatorname{cen} U$ is dense in U for each $U \in \mathsf{ClopUp}(X)$.

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Lemma

L is a zero-dimensional frame iff X is a zero-dimensional L-space. Either implies Y is a zero-dimensional space. The converse holds if Y is dense in X (i.e. L is spatial).

Stone L-spaces

Definition

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2. Let StoneLPries be the category of Stone L-spaces and L-morphisms.

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Theorem

StoneLPries *is equivalent to* Stone *and dually equivalent to* StoneFrm.

AlgFrm $\leftrightsquigarrow$ AlgLPries $\longleftrightarrow$ KBSob

$$\begin{array}{ccccc} \mathsf{AlgFrm} & \leftrightsquigarrow & \mathsf{AlgLPries} & \longleftrightarrow & \mathsf{KBSob} \\ \vee\!\!\!/\!/ & & \vee\!\!\!/\!/ & & \vee\!\!\!/\!/ \\ \mathsf{AriFrm} & \leftrightsquigarrow & \mathsf{AriLPries} & \longleftrightarrow & \mathsf{SKBSp} \end{array}$$

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\begin{array}{ccccc}
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\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
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\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
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$$

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\rotatebox{-90}{$\subsetneq$} & & \rotatebox{-90}{$\subsetneq$} & & \rotatebox{-90}{$\subsetneq$} \\
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Thank you!

Categories

Category	Objects	Morphisms
AlgFrm	algebraic frames	coherent frame homomorphisms
AriFrm	arithmetic frames	coherent frame homomorphisms
CohFrm	coherent frames	coherent frame homomorphisms
StoneFrm	Stone frames	frame homomorphisms
KBSob	compactly based sober spaces	coherent maps
SKBSp	stably compactly based spaces	coherent maps
Spec	spectral spaces	coherent maps
Stone	Stone spaces	continuous maps
AlgLPries	algebraic L-spaces	coherent L-morphisms
AriLPries	arithmetic L-spaces	coherent L-morphisms
CohLPries	coherent L-spaces	coherent L-morphisms
StoneLPries	Stone L-spaces	L-morphisms

Connection to Priestley and Stone duality

Let L be a coherent frame, X_L its Priestley space, and Y_L the spatial part of X_L.

Then the collection $K(L)$ of compact elements is a bounded distributive lattice, and the poset of prime filters $X_{K(L)}$ is isomorphic to $(Y_L, \subseteq)$.

However, the Priestley topology of $X_{K(L)}$ is not the same as the topology of X_L restricted to Y_L.

The topology on Y_L corresponding to $X_{K(L)}$ is generated by the basis $\{(U \setminus V) \cap Y_L \mid U, V \in \mathsf{ClopSUp}(X_L)\}$.

Similarly, if L is a Stone frame then $K(L)$ is a boolean algebra whose Stone dual $X_{K(L)}$ corresponds to Y_L with the topology generated by $\{U \cap Y_L \mid U \in \mathsf{ClopBi}(X_L)\}$.

Complemented being exactly compact elements

Recall, if L is a Stone frame, then an element is complemented iff it is compact.

In the language of Priestley duality we get:

Lemma

Let X be a Stone L-space. Then

- $\mathsf{ClopSUp}(X) = \mathsf{ClopBi}(X)$.
- $\mathrm{core}\, U = \mathrm{cen}\, U$ *for each* $U \in \mathsf{ClopUp}(X)$.

Morever, we have the following more general observations:

Lemma

1. *If X is L-compact, then every closed biset is a Scott upset.*
2. *If X is L-regular, then every Scott upset is a biset.*