Lax and steady wins the race

Canonical Formulas for the Lax Logic

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Canonical formulas

Zakharyaschev (1989; 1991) introduced canonical formulas as a method to uniformly axiomatise all intermediate and transitive modal logics.

This method provided a lot of structure in the study of intermediate logics, for example:

- Simple instances of canonical formulas characterise subframe and cofinal subframe logics.
- Zakharyaschev obtained a proof for the Dummett-Lemmon conjecture that the least modal companion of a Kripke-complete intermediate logic is Kripke-complete.

The method of canonical formulas

Essentially, the method of canonical formulas is a two-step procedure.

1. Characterise every formula ϕ with a finite number of refutation patterns.
   - This is a finite collection of counter-models $A_1, \ldots, A_n$ with some parameters $D_1, \ldots, D_n$.
   - We obtain them by the local finiteness of some suitable reduct.
2. Encode the refutation patterns into formulas.
   - Similar to the construction of Jankov formulas.

Then every formula is semantically equivalent to some conjunction of canonical formulas. Consequently, they axiomatise every logic.

Limitations of canonical formulas

Zakharyaschev's approach to canonical formulas relies on the dual structure of finitely generated Heyting and K4-algebras.

Consequently, is has only been applied to intermediate and transitive modal logics.

This thesis is the first step to extending the method of canonical formulas to the domain of intuitionistic modal logics.

The aim of this thesis is to apply the method of canonical formulas to the Lax Logic and obtain some similar results as mentioned before.

Lax Logic

A unary operator $\Box$ is a lax modality iff it respects the axiom:

$$p \to \Box q \leftrightarrow \Box p \to \Box q$$

or equivalently:

$$\Box(p \to q) \to (\Box p \to \Box q) \quad \text{and} \quad (\Box\Box p \vee p) \to \Box p$$

A lax logic is an intermediate logic extended with a lax modality, i.e., an intuitionistic modal logic that validates the axioms above.

Lax semantics

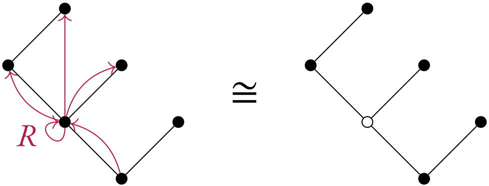
Algebraic semantics for lax logics are given by nuclear algebras – Heyting algebras with nuclei/lax modalities.

A lax (Kripke) frame is a tuple $(X, \leq, R)$ such that $\leq$ is a partial order and:

$$\leq \circ R \circ \leq = R, \qquad R \subseteq \leq, \qquad R \subseteq R^2.$$

The Kripke semantics ($\Vdash$) is determined by the expected clauses, for example: $X, x \Vdash \Box\phi$ iff xRy entails $X, y \Vdash \phi$.

G. Bezhanishvili and Ghilardi (2007) showed that lax relations are determined by their reflexive points.



Lax Canonical Formulas

We want to apply the method of canonical formulas to the lax case.

1. We require a locally finite reduct: G. Bezhanishvili, N. Bezhanishvili, Carai, Gabelaia, Ghilardi, and Jibladze (2020) showed that the $\vee$-free reduct of nuclear algebras is locally finite.
2. We can use the general algebraic method.
   - With each finite (s.i.) nuclear algebra A we associate formulas $\alpha(A, D_\vee, D_\Box, D_\bot)$ encoding some structure of A.
   - The exact construction of such formulas is not important. What actually matters is the refutation criterion:

Theorem

$B \nvDash \alpha(A, D_\vee, D_\Box, D_\bot)$ *iff there is a homomorphic image* C *of* B *and an implicative semilattice embedding from* A *into* C *that respects the parameters.*

Canonical axiomatisations

A lax canonical formula is an algebra-based formula $\alpha(A, D_\vee, \Box, \bot)$ that encodes the complete $\vee$-free reduct of the algebra.

Theorem

All lax logics are axiomatised by lax canonical formulas.

More generally:

Theorem

Given $F \subseteq \{\vee, \Box, \bot\}$. If a lax logic is axiomatised by F-free formulas, then it is axiomatised by algebra-based formulas such that $D_f = \emptyset$ for $f \in F$.

Esakia duality and beyond

Esakia duality is an order-topological representation of Heyting algebras and their homomorphisms.

G. Bezhanishvili and Ghilardi (2007) established nuclear Esakia duality – extending Esakia duality for nuclear algebras. The extended Esakia spaces are called nuclear spaces.

G. Bezhanishvili and N. Bezhanishvili (2009) generalised Esakia duality to account for $\{\wedge, \rightarrow\}$-homomorphisms between Heyting algebras.

- Dually, these homomorphisms correspond to partial Esakia morphisms – partial continuous p-morphisms.

To account for lax canonical formulas we need to extend this duality to the nuclear case.

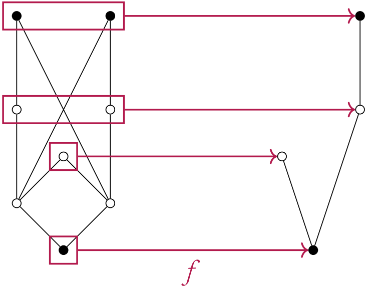
- This gives us partial Esakia morphisms that also preserve the lax relation: partial nuclear morphisms.

A problem with lax subframes

Consider the two frames on the right.

f is a partial nuclear morphism:

1. $f[\uparrow x] = \uparrow fx$ for all $x \in \mathsf{dom}(f)$.
2. $f[R[x]] = R[f[\uparrow x]]$ for all $x \in X$.



We expect $\mathsf{dom}(f)$ to be a subframe but the frame on the left validates $\alpha(\mathsf{dom}(f), \Box, \bot)$.

Consequently, the logic axiomatised by this formula is not closed under finite domains of partial nuclear morphisms.

The problem is that the second condition on partial nuclear morphisms reaches outside of the domain.

Steady subframes

The solution is to focus only on one inclusion of the condition:

$$R[f[\uparrow x]] \subseteq f[R[x]] \text{ for all } x \in X. \qquad \text{(Steadiness)}$$

This is equivalent to $R[fx] \subseteq f[R[x]]$ for all $x \in \mathsf{dom}(f)$. Hence, it stays inside the domain.

This inspires the following lax definition for subframes:

$(Y, \leq_Y, R_Y)$ is a steady subframe of $(X, \leq_X, R_X)$ iff

- $Y \subseteq X$;
- $\leq_Y$ is the restriction of $\leq_X$ to Y;
- $y \in Y$ is a reflexive point of R_Y iff y is a reflexive point of R_X.

Finite domains of steady morphisms are steady subframes!

Steady logics and steady canonical formulas

A steady canonical formula $\beta(A, D_\vee, D_\Box, \bot)$ is an algebra-based formula that encodes $\Box$ in the steady direction and the other direction only for $D_\Box$.

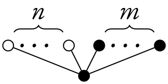
Lax canonical formulas are special instances ($D_\Box = A$) of steady canonical formulas. Hence, steady canonical formulas axiomatise all lax logics.

A logic is steady iff it is axiomatisable by steady canonical formulas of the form $\beta(A, \bot) = \beta(A, \emptyset, \emptyset, \bot)$.

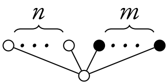
Theorem

1. *A logic is steady iff it is the logic of a class of lax frames closed under steady subframes.*
2. *All steady logics have the fmp.*

Examples of steady logics



$\mathfrak{F}_{\bullet}^{n,m}$



$\mathfrak{F}_{\circ}^{n,m}$

$n + m \geq 2$

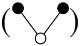
Theorem

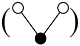
1. $PLL \oplus \beta(\mathfrak{F}_{\bullet}^{n,m})$ *is the logic of all finite rooted frames that contain no antichain of* $n + m$ *elements with at least n nuclear elements.*
2. $PLL \oplus \beta(\mathfrak{F}_{\bullet}^{n,m}, \bot)$ *is the logic of all finite rooted frames that do not have* $n + m$ *maximal elements with at least n nuclear.*
3. $PLL \oplus \beta(\mathfrak{F}_{\circ}^{n,m})$ *is the logic of all finite nuclear* $\circ$*-rooted frames that contain no antichain of* $n + m$ *elements with at least n nuclear elements.*
4. $PLL \oplus \beta(\mathfrak{F}_{\circ}^{n,m}, \bot)$ *is the logic of all finite* $\circ$*-rooted frames that do not have* $n + m$ *maximal elements with at least n nuclear.*

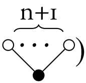
More examples of steady logics

LMx $=$ $\mathsf{PLL} \oplus \beta($  $)$

LRt $=$ $\mathsf{PLL} \oplus \beta($  $)$

LIC $=$ $\mathsf{PLL} \oplus \beta($  $)$

LLn $=$ $\mathsf{PLL} \oplus \beta($  $)$

BIW_n $=$ $\mathsf{PLL} \oplus \beta($ 

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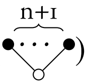
BRW_n $=$ $\mathsf{PLL} \oplus \beta($ 

$)$

Dummett-Lemmon conjecture

We will prove a lax Dummett-Lemmon conjecture.

Theorem

If $\mathsf{L} = \mathit{IPC} \oplus \Gamma$ *is Kripke-complete then* $\mathsf{L}^{\bullet} = \mathit{PLL} \oplus \Gamma$ *is Kripke-complete.*

Essentially, we need two procedures:

1. Find $\psi \notin \mathsf{L}$ for every $\phi \notin \mathsf{L}^{\bullet}$.
2. Extend an L-frame $X \nVdash \psi$ to an $\mathsf{L}^{\bullet}$-frame $X' \nVdash \phi$.

The former procedure is easily found using steady canonical formulas.

Lemma 1.

If $\mathsf{L} \vdash \alpha(A, D_{\vee}, \bot)$ then $\mathsf{L}^{\bullet} \vdash \beta(A, D_{\vee}, D_{\Box}, \bot)$.

Lemma 2.

Let A be a finite nuclear algebra and $(X, \leq)$ a Kripke frame. If $h : A \to \mathsf{Up}(X)$ is an $\{\wedge, \to\}$-embedding then there exists a lax relation R on X such that h is nuclear.

Proof sketch. Let X' and Y be the dual spaces of $\mathsf{Up}(X)$ and A. We can assume $X \subseteq X'$.

By duality, we have an onto partial Esakia morphism $f : X' \to Y$.

Define R on X by setting its reflexive points to the preimage of the R-reflexive points of Y.

Then $f : X \to Y$ is a nuclear subreduction. The dual of f is the required homomorphism. ∎

Preservation results

Theorem

If L is Kripke-complete then $\mathsf{L}^\bullet$ is Kripke-complete.

Proof. Suppose $\mathsf{L}^\bullet \nvdash \beta(A, D_\vee, D_\Box, \bot)$. By Lemma 1, $\mathsf{L} \nvdash \alpha(A, D_\vee, \bot)$. Then there exists an L-frame $X \nVdash \alpha(A, D_\vee, \bot)$. By Lemma 2, we can define R on X such that $(X, R) \nVdash \beta(A, D_\vee, D_\Box, \bot)$. ■

Steady canonical formulas characterise the structure of $\mathsf{L}^\bullet$ perfectly, i.e., they are an important ingredient of this proof.

Other properties preserved in $\mathsf{L}^\bullet$ are fmp, tabularity, decidability (if Kripke-complete).

Conclusion

Lax canonical formulas and steady canonical formulas can be used to axiomatise all lax logics.

However, steady canonical formulas describe subtleties of the structure of lax logics in a clearer manner:

- They characterise steady logics – a class of lax logics with good properties.
- We can use them to obtain a proof for a lax Dummett-Lemmon conjecture.

Besides, it seems unfeasible to generalise lax canonical formulas to other intuitionistic modal logics since they heavily rely on the local finiteness of the $\{\Box, \wedge, \rightarrow\}$-reduct.

Steady canonical formulas on the other hand do not strictly make use of this reduct.

Future work

Axiomatise logics extending IS4 with "co-steady" canonical formulas.

Preservation results for less simple translations of intermediate logics into lax logics.

Investigating "semantic" translations.

Admissible rules for lax logics.

Thank you!

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