Priestley duality and Pointfree topology

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An image

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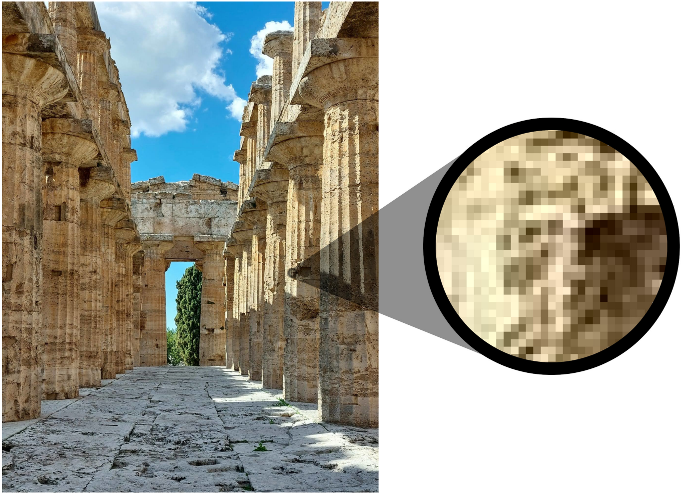
An image



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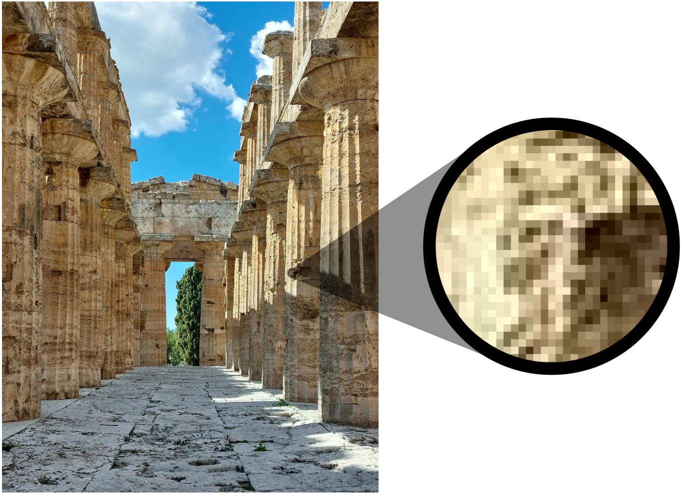

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Would you start by listing the individual pixels?

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However, despite its generality pointfree topology often loses some of the geometric intuition tied to points.

Goal of this talk: to show how **Priestley duality** can recover this intuition by representing frames as ordered topological spaces.

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Hausdorff introduced the concept of a *neighborhood*, shifting attention from metric ideas to general spatial structure.

Subsequent work by Kuratowski, Alexandroff, Urysohn, and Sierpiński established the abstract foundations of topology as we know it today.

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Building on this, researchers such as Dowker & Papert-Strauss, Isbell, Banaschewski, and Johnstone established the modern theory of pointfree topology.

Spatiality theorems

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Such results connect classical and pointfree topology by identifying the frames that can be interpreted in familiar topological terms.

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How can we bridge this gap and recover a spatial understanding of frames?

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This duality specializes to frames, offering a powerful tool to study both spatial and non-spatial frames through order and topology.

It thus restores the geometric intuition that pointfree topology had set aside.

Priestley duality

A Priestley space is a compact topological space X equipped with a partial order $\leq$ satisfying

$$x \not\leq y \implies \exists U \in \mathrm{ClopUp}(X)\,(x \in U \text{ and } y \notin U),$$

where $\mathrm{ClopUp}(X)$ denotes the clopen upsets of X.

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Let Pries be the category of Priestley spaces and continuous order-preserving maps, and DLat the category of bounded distributive lattices and bounded lattice homomorphisms.

Theorem (Priestley, 1970)

Pries *and* DLat *are dually equivalent.*

Priestley duality: some details

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- For every Priestley space X, the set $\mathrm{ClopUp}(X)$ of clopen upsets forms a bounded distributive lattice.
- For every bounded distributive lattice D, its Priestley space X_D is the set of prime filters of D, ordered by inclusion and topologized by the basis

$$\{\sigma(a) \cap \sigma(b)^c \mid a, b \in D\},$$

  where $\sigma(a) = \{x \in X_D \mid a \in x\}$ is the Stone map.

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The map $\sigma : D \to \mathrm{ClopUp}(X_D)$ is a bounded lattice isomorphism, serving as the unit of the duality. For instance, $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b)$ and $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$.

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By compactness, finitely many such sets suffice, so there exist clopen V and D with $x \in V \cap D \subseteq U$. Hence X is zero-dimensional. (In fact, it has a basis of convex clopens.) □

Closed sets in Priestley spaces

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(1) *Every closed upset (downset) is an intersection of clopen upsets (downsets).*

(2) *The upset (downset) of a closed set is closed.*

(3) *For every $x \in F$ with F closed, there exists $y \in \min F$ such that $y \leq x$.*

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For (2), one shows that $\uparrow F = \bigcap\{U \in \mathrm{ClopUp}(X) \mid F \subseteq U\}$, and (3) uses Zorn's Lemma. □

Priestley spaces of frames

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The key observation is that frames are (special) bounded distributive lattices, so we can apply Priestley duality to them.

In particular, we need to describe the Priestley spaces of complete lattices where each join satisfies ($\blacklozenge$).

Joins in Priestley duality

Lemma

Let D be a bounded distributive lattice and $S \subseteq D$. Then

(1) $\bigvee S$ *exists iff* $\uparrow \mathrm{cl} \bigcup \sigma[S]$ *is a clopen upset.*

(2) $\bigvee S$ *exists and satisfies* ($\blacklozenge$) *for all* $a \in L$ *iff* $\mathrm{cl} \bigcup \sigma[S]$ *is a clopen upset.*

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Proof sketch.

(1) The direction ($\Leftarrow$) follows since $\uparrow \mathrm{cl} \bigcup \sigma[S]$ is then the least upper bound of $\sigma[S]$ in $\mathrm{ClopUp}(X)$. Conversely, if $\bigvee S$ exists, one checks that $\sigma(\bigvee S) = \uparrow \mathrm{cl} \bigcup \sigma[S]$.

(2) By (1), exactness of $\bigvee S$ amounts to $\mathrm{cl} \bigcup \sigma[S]$ being an upset. If it is, the distributivity law (♦) holds in $\mathrm{ClopUp}(X)$; if not, one finds $x \in \uparrow \mathrm{cl} \bigcup \sigma[S] \setminus \mathrm{cl} \bigcup \sigma[S]$, and compactness gives clopen U, V witnessing the failure of (♦). □

Wigner–Pultr–Sichler duality

Theorem (Wigner, 1979)

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We will now describe three key notions of pointfree topology in terms of Priestley spaces: spatiality, subfitness, and compactness.

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We now define a class of points in the Priestley space of a frame that will allow us to characterize spatiality.

Definition

Let X be the Priestley space of a frame.

1. A point $x \in X$ is called localic if ${\downarrow}x$ is open.
2. Let $\mathrm{loc}X$ denote the set of all localic points of X.

Localic points and completely prime filters

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($\Rightarrow$) Suppose x is a completely prime filter. By Priestley properties, $\downarrow x$ is closed, so let $U = (\downarrow x)^c$, an open upset.

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This identification will allow us to express spatiality directly in terms of the Priestley space.

Spatiality

Theorem

If L is a frame and X its Priestley space, then L is spatial iff $\mathrm{loc}X$ is dense in X.

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($\Leftarrow$) Suppose $\mathrm{loc}X$ is dense. If $a \nleq b$, then $\sigma(a) \cap \sigma(b)^c \neq \varnothing$. Thus there exists $y \in \mathrm{loc}X \cap \sigma(a) \cap \sigma(b)^c$, so $a \in y$ but $b \notin y$.

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Separation axioms

Higher separation axioms, such as regularity and normality, can be expressed directly in terms of the frame of opens, and hence extend naturally to the pointfree setting.

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Example. If X is a T_1-space, then $\Omega(X)$ is subfit.

Subfitness

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$$\bigcap \{D \in \mathrm{ClopDn}(X) \mid y \in D\} \subseteq \sigma(a).$$

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$$\bigcap \{D \in \mathrm{ClopDn}(X) \mid y \in D\} \subseteq \sigma(a).$$

By compactness, there exists a clopen downset $D = \sigma(c)^c \subseteq \sigma(a)$ such that $y \in \sigma(c)^c$. Then $\sigma(a) \cup \sigma(c) = X$ and $\sigma(b) \cup \sigma(c) \neq X$, as required. □

Compactness

Compactness plays a central role in both classical and pointfree topology, as many structural results depend on it.

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In the classical setting, compactness can be expressed purely in terms of the frame of opens, making it one of the easiest topological notions to generalize.

A frame L is compact if $\bigvee U = 1$ implies there exists a finite subset $V \subseteq U$ such that $\bigvee V = 1$.

Compactness and Priestley spaces

Theorem

If L is a frame and X its Priestley space, then L is compact iff $\min X \subseteq \operatorname{loc} X$.

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Compactness and Priestley spaces

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If L is a frame and X its Priestley space, then L is compact iff $\min X \subseteq \operatorname{loc} X$.

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Isbell's Spatiality Theorem

Let L be a frame and X its Priestley space.

1. L is spatial iff $\mathrm{loc}X$ is dense in X.
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Suppose L is compact and subfit. Since L is subfit, $\min X$ is dense, and since L is compact, $\min X \subseteq \mathrm{loc}X$. Hence $\mathrm{loc}X$ is dense, and therefore L is spatial. □

Conclusion

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Together these results show that Priestley duality for frames offers a geometrically meaningful framework for studying pointfree topology.


Thank you!