Generalizing the d-nucleus to stably continuous frames

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Topology, Algebra, and Categories in Logic

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Joint work with G. Bezhanishvili, P. Bhattacharjee, and M. A. Moshier

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In a sense, the d-nucleus captures the part of double negation determined by *compact approximations*. The generalized $\underline{d}$-nucleus replaces these compact approximations by *way-below approximations*.

Double negation in topology

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If X is compact Hausdorff, the Stone space of $\mathrm{RO}(X)$ is the Gleason cover of X, that is, its projective cover. It is extremally disconnected, and the canonical surjection onto X is irreducible.

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Thus, there is a natural embedding $X \longrightarrow \mathsf{spec}(X)$ given by $x \longmapsto \mathcal{N}_x$.

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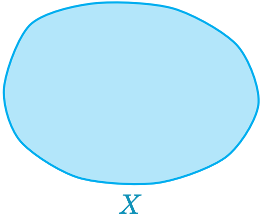

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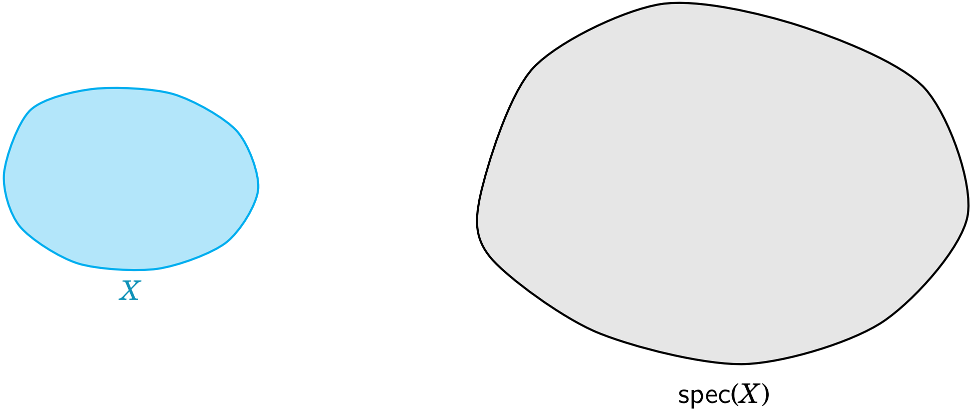

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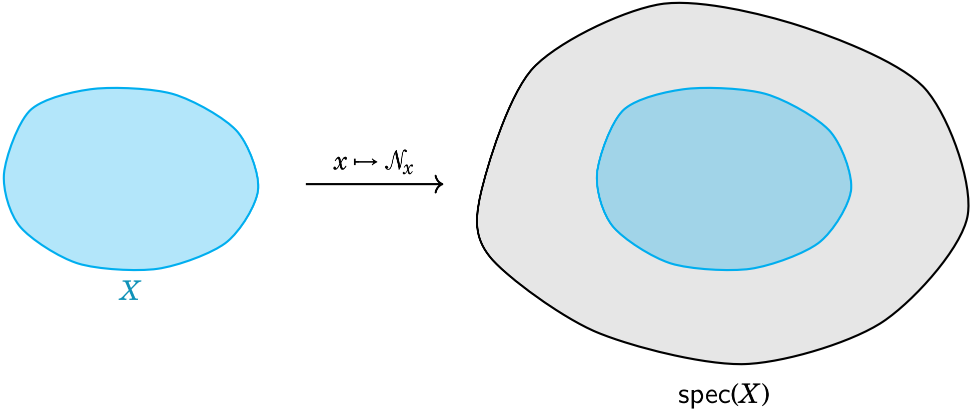

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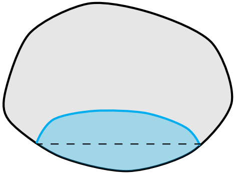

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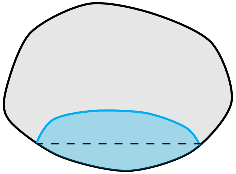

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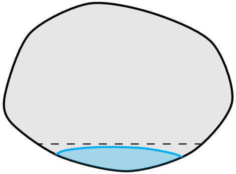

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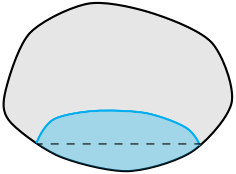

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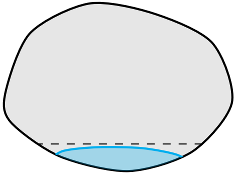

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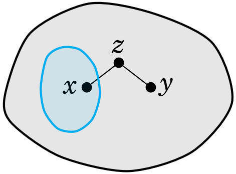

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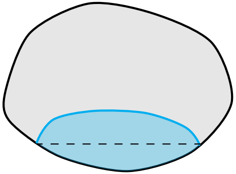

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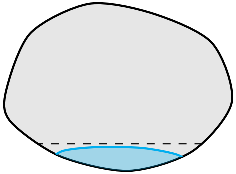

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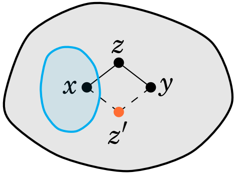

Diamond property.

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In short, the restriction $g : \max \mathsf{spec}(X) \to X$ realizes the Gleason cover.

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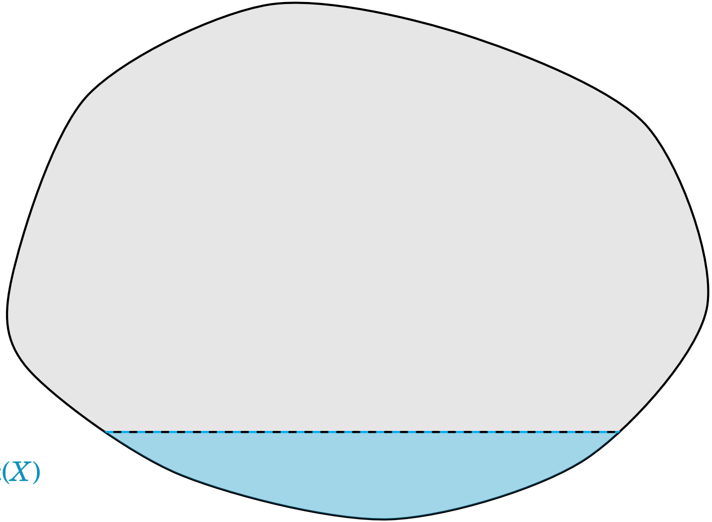

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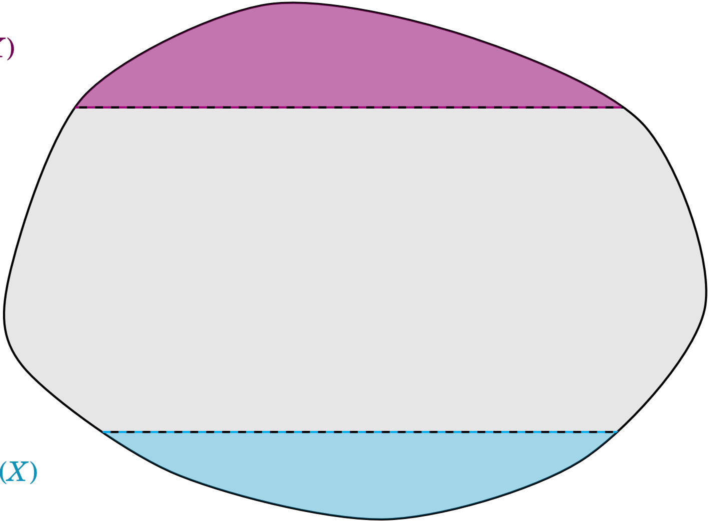

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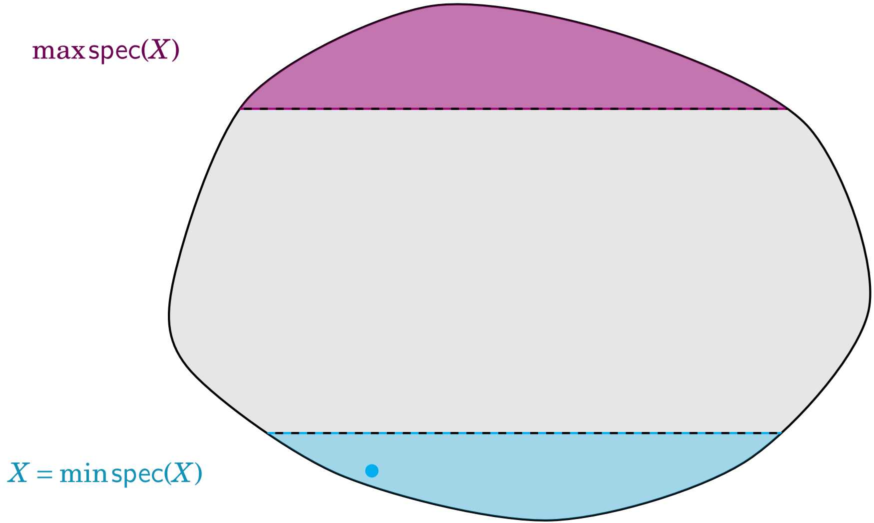

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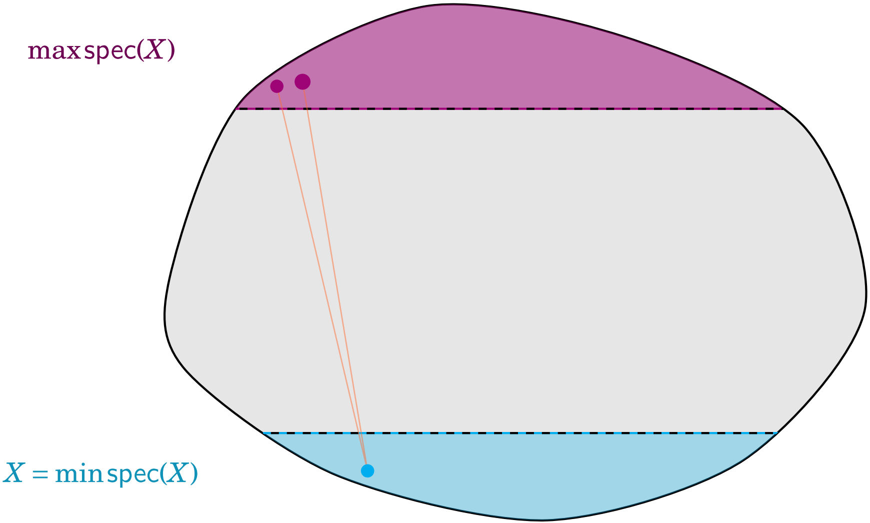

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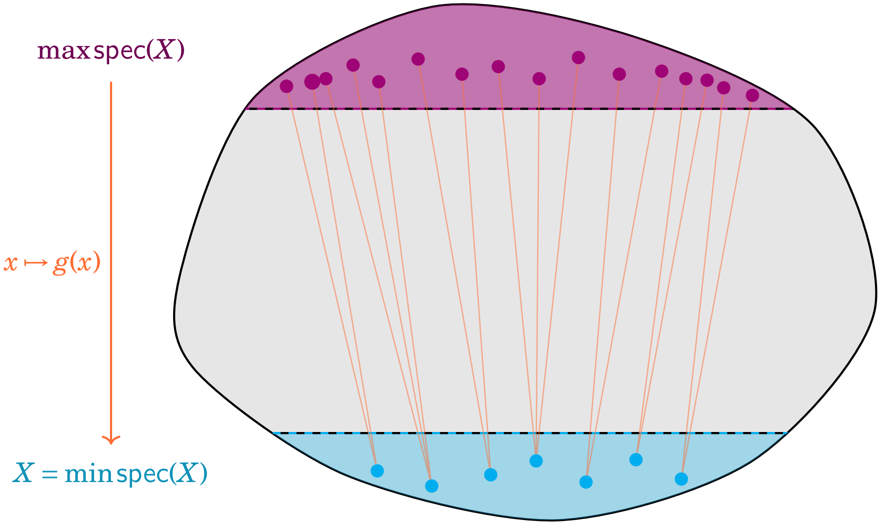

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Nuclei correspond order-reversingly, via their images, to sublocales. These are the pointfree analogues of subspaces. Thus, Isbell's theorem says that the booleanization $\mathsf{B}(L)$ is the smallest dense sublocale of L.

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for every directed $D \subseteq L$. (Every finite subset of D has an upper bound in D.) Equivalently, the corresponding sublocale is closed under directed joins.

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Theorem (Escardó, 1998)

Every sober space has a smallest dense Scott continuous subspace.

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Martínez and Zenk showed that, on every arithmetic frame L, the formula

$$da = \bigvee \{\neg\neg k \mid k \text{ is compact and } k \leq a\}$$

defines a nucleus, called the d-nucleus. For the frame of ℓ-ideals, the fixpoints of d are precisely the d-ideals.

From arithmetic to stably continuous frames

For $b,a \in L$, we say b is way below a, and write $b \ll a$, if whenever $a \leq \bigvee D$ for a directed set $D \subseteq L$, there is some $d \in D$ with $b \leq d$.

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Passing from algebraic frames to continuous frames amounts to replacing approximation by compact elements with approximation via the way-below relation.

This suggests replacing the compact elements in the definition of the d-nucleus by the elements way below a.

(Recall $da = \bigvee\{\neg\neg k \mid k \text{ is compact and } k \leq a\}$.)

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Lemma (Escardó, 1998)

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Applied to double negation nucleus, this gives

$$\underline{d}a := \underline{\neg\neg} a = \bigvee \{ \neg\neg b \mid b \ll a \}.$$

Thus, $\underline{d}$ is the largest Scott continuous nucleus below double negation. On algebraic frames, *way-below approximation* reduces to approximation by compact elements, so $\underline{d} = d$.

The $\underline{d}$-subspace in Priestley duality

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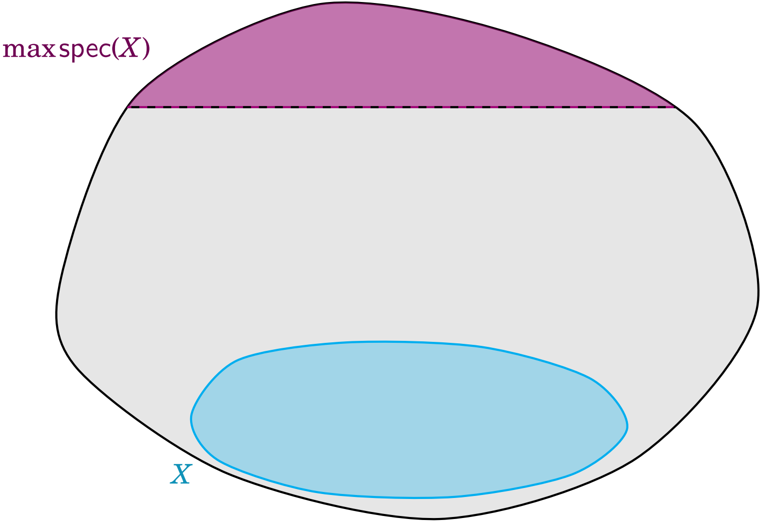
Theorem ([BBMM26])

The subspace of X corresponding to the $\underline{d}$-nucleus is

$$X_{\underline{d}} = \{x \in X \mid x \textit{ is the greatest point of } X \textit{ below some } y \in \max \mathsf{spec}(X)\}.$$

Consequently, $X_{\underline{d}}$ is the smallest dense Scott continuous subspace of X.

(Recall that the booleanization corresponds to $\max \mathsf{spec}(X)$.)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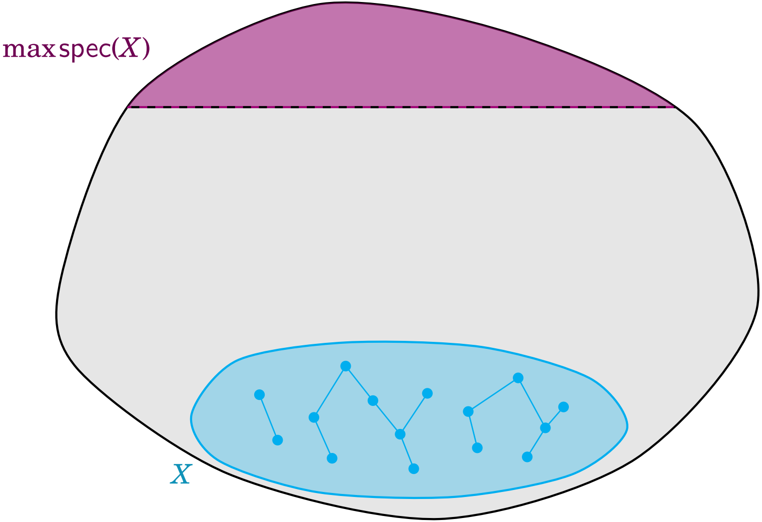

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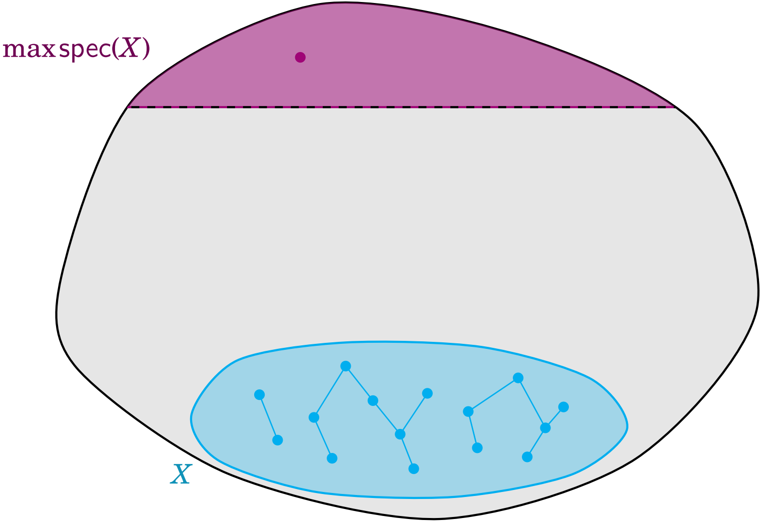
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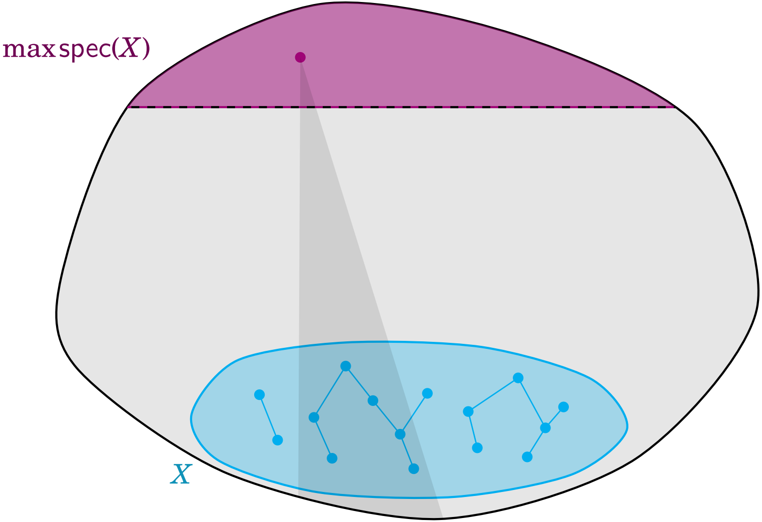

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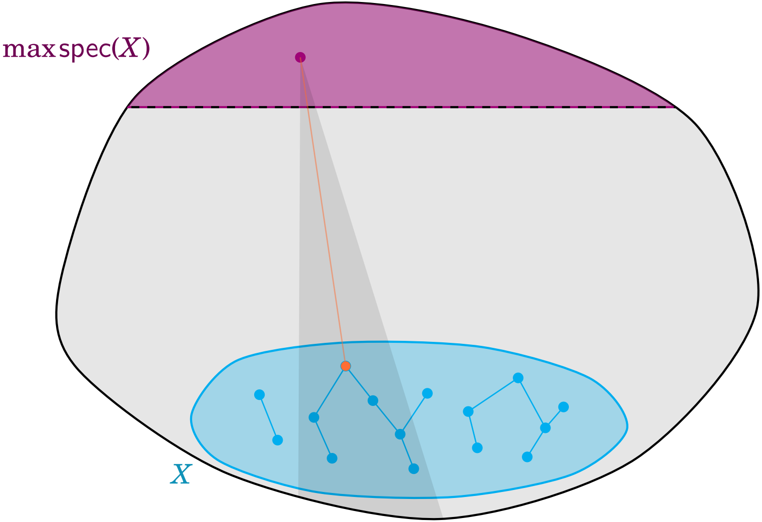

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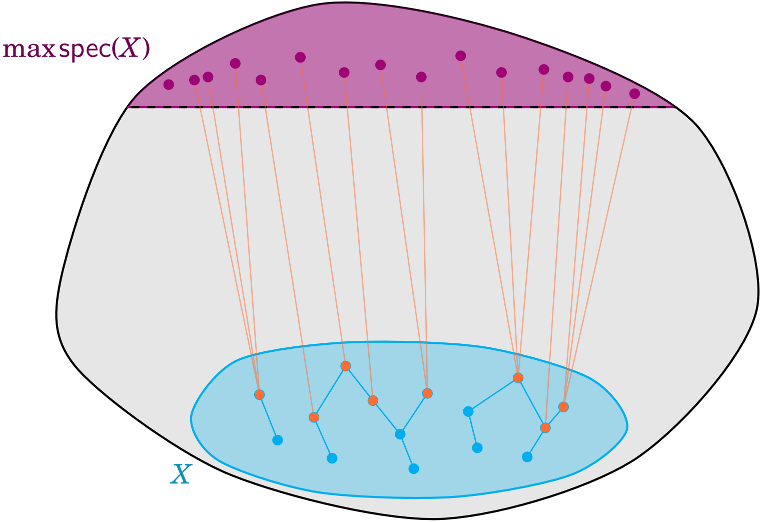
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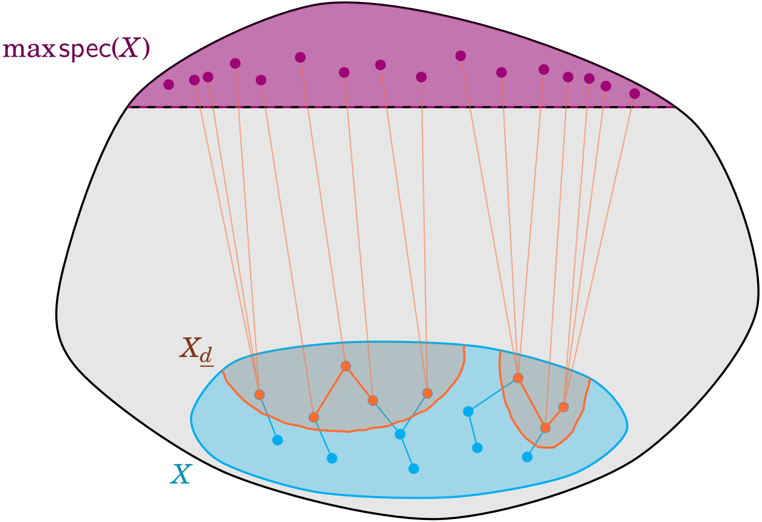
max spec(X)
X_d
X

Maximal d-spectra

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For the d-nucleus on the frame of ℓ-ideals of a Riesz space, this recovers the *space of maximal d-ideals* studied by Huijsmans and de Pagter. Bhattacharjee later introduced the maximal d-spectrum of an arbitrary arithmetic frame.

Iterating the Priestley construction

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We can now apply the Priestley description of the $\underline{d}$-subspace of $\mathsf{spec}(X)$, yielding that

$$\mathsf{spec}(X)_{\underline{d}} = \{x \in \mathsf{spec}(X) \mid x \text{ is the greatest point below some } y \in \max \mathsf{spec}^2(X)\}$$

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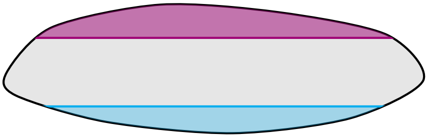
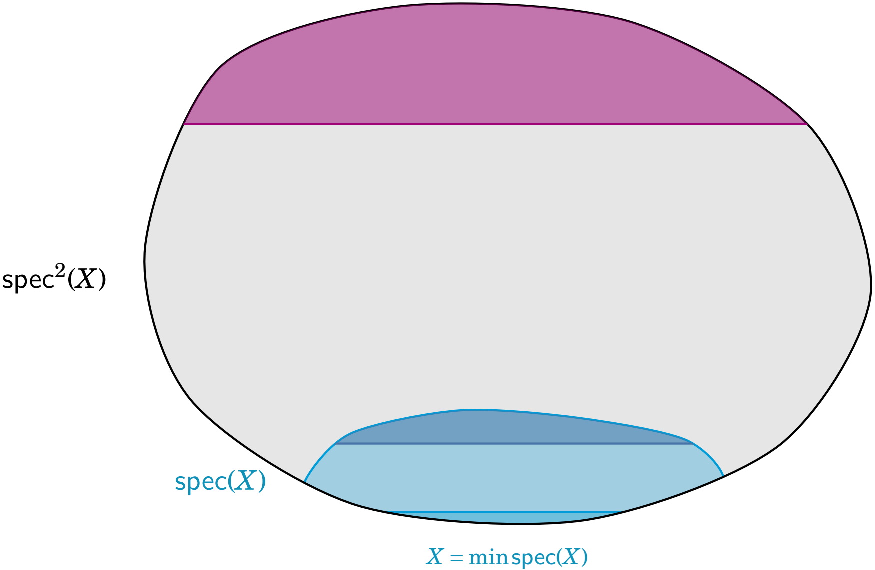
This subspace is another realization of the Gleason cover, as the next slide illustrates.

spec(X)



$X = \min \operatorname{spec}(X)$



$\mathrm{spec}^2(X)$
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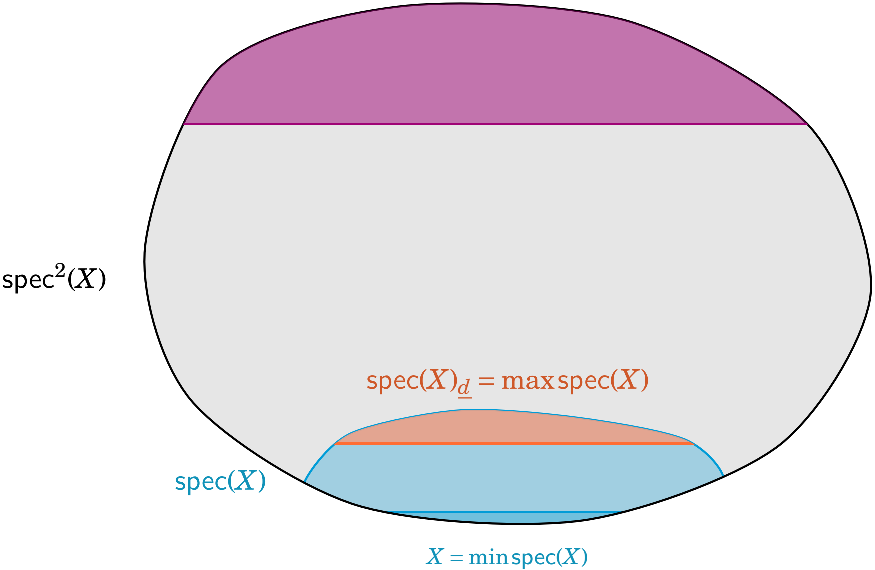

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The Gleason cover as a maximal d-spectrum

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Since maximal $\underline{d}$-elements of the frame of ideals are maximal d-ideals, we obtain the following:

Theorem ([BBMM26])

The Gleason cover of a compact Hausdorff space X is homeomorphic to the space of maximal d-ideals of $\Omega(X)$.

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The Gleason cover of a compact Hausdorff space X is homeomorphic to the space of maximal d-ideals of $\Omega(X)$.

Consequently, every extremally disconnected compact Hausdorff space is realized as a maximal d-spectrum.

Units and compactness

Maximal d-spectra of arithmetic frames are always T_1, but they need not be compact. A standard additional assumption is that the frame has a unit, that is, a compact dense element. Here dense means that $\neg e = 0$.

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We therefore call $e \in L$ a Scott unit if $\{a \in L \mid e \ll a\}$ is a Scott-open filter consisting of dense elements. On arithmetic frames, the existence of a Scott unit is equivalent to the existence of a unit.

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Theorem ([BBMM26])

A stably continuous frame L has a Scott unit iff $\underline{d}L$ is max-bounded and $\max(\underline{d}L)$ is compact.

Here, $\underline{d}L$ is max-bounded if every proper $\underline{d}$-element lies below a maximal $\underline{d}$-element.

The maximal d-spectrum problem

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Every locally compact Hausdorff space is homeomorphic to the maximal $\underline{d}$-spectrum of a continuous regular frame. In particular, every compact Hausdorff space is homeomorphic to the maximal $\underline{d}$-spectrum of a compact regular frame.

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The key observation is that $\underline{d}$ is the identity on every continuous regular frame. Moreover, for a continuous regular frame, the maximal spectrum agrees with its space of points.

The arithmetic case

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Corollary ([BBMM26])

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This solves the representation problem in the zero-dimensional case.

Thanks!

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