# Locally compact and sober, but not quite enough points


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Seminar on Domain Theory and its Applications

Tianyuan Mathematics Research Center, Yunnan, China

December 19, 2025

# 介绍
# Introduction

# Motivation

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We examine the classic result in this new setting.

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**Example**

The lattice of open subsets $\Omega(X)$ of a locally compact space $X$ is a continuous frame.

# Hofmann–Lawson duality

In fact, all continuous frames are of this form:

**Theorem (Hofmann–Lawson[1])**

*The category of continuous frames is dually equivalent to the category of locally compact sober spaces. In particular, every continuous frame $L$ is isomorphic to $\Omega(X)$ for some locally compact sober space $X$.*

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A frame is called **spatial** if it is isomorphic to the opens of a topological space.

**Corollary**

*Every continuous frame is spatial.*

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# Pointfree topology

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This shift allows for:

- More algebraic reasoning about topological spaces.
- More constructive treatments of results (e.g., Tychonoff's theorem[2] or the existence of Čech–Stone compactifications[3]).

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The common approach is to study frames[4], i.e., generalized lattices of open sets.

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In this way, a topological space can be recovered from its frame of opens iff every completely prime filter of that frame corresponds to a unique point of the space. Such spaces are called **sober**.

# The adjunction

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On the one hand, this shows that frames faithfully generalize sober spaces.

On the other, many spaces lie outside this picture since they are not sober.

# 替代方法
# Alternative approach

Frame theory is not the only pointfree approach to topology.

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Instead of abstracting lattices of the form $\Omega(X)$ to frames, one can abstract powerset lattices $\mathscr{P}(X)$ (equipped with their topological interior) to complete interior algebras.

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This alternative, interior-based approach began with Kuratowski's closure axioms[5] and was further generalized by McKinsey and Tarski.[6]

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Although it became central in modal logic, it was largely overlooked in pointfree topology, but recent work reintroduced McKinsey–Tarski (MT) algebras into the pointfree study of spaces:

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# Literature of this talk


- Ranjitha R. "McKinsey-Tarski algebras: an alternative pointfree approach to topology". PhD thesis. New Mexico State University, 2025
  - G. Bezhanishvili and Ranjitha R. "McKinsey-Tarski algebras: an alternative pointfree approach to topology". In: *Topology Appl.* 339 (2023), Paper No. 108689
  - G. Bezhanishvili and Ranjitha R. "Local Compactness in MT-Algebras". In: *Topology Proc.* 66 (2025), pp. 15–48
- G. Bezhanishvili, Ranjitha R., A. L. Suarez, and J. Walters-Wayland. "The Funayama envelope as the $T_D$-hull of a frame". In: *Theory Appl. Categ.* 44 (2025), pp. 1106–1147
- G. Bezhanishvili, S. D. Melzer, Ranjitha R., and A. L. Suarez. "Local compactness does not always imply spatiality". In: *Q&A in Gen. Top.* (2026). To appear.

# MT-algebras

## Definition

An **MT-algebra** $M$ is a complete boolean algebra equipped with an interior operator $\Box$, i.e.,

$$\Box 1 = 1, \quad \Box(a \wedge b) = \Box a \wedge \Box b, \quad \Box\Box a = \Box a, \quad \text{and } \Box a \leq a$$

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**Example**

1. For each topological space $X$, the powerset $\mathscr{P}(X)$ (equipped with the topological interior) forms an MT-algebra.
2. For every frame $L$, the booleanization $N(L)_{\neg\neg}$ of the frame of nuclei is an MT-algebra. The box is determined by the embedding of $L$ into $N(L)$.

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We think of the atoms $\mathcal{at}(M)$ as the **points** of an MT-algebra $M$. The sets of the form $\{x \in \mathcal{at}(M) \mid x \leq \Box a\}$ for $a \in M$ is a topology on $\mathcal{at}(M)$.

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No separation assumption on $X$ is needed to recover it from its MT-algebra. Compare this to the situation with frames, where a space can only be recovered from its frame when it is sober.

# The MT-adjunction

**Theorem ([BR23])**

1. *The assignments* $X \mapsto \mathscr{P}(X)$ *and* $M \mapsto \mathcal{A}t(M)$ *form an adjunction between* **Top** *and the category* **MT** *of MT-algebras.*

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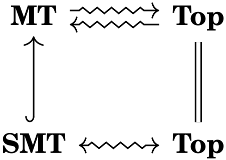
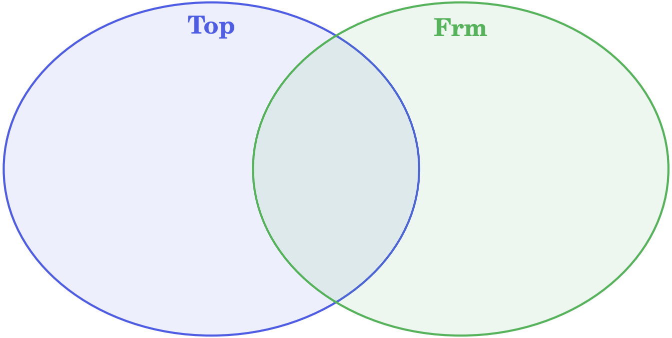



In this sense, MT-algebras generalize all spaces.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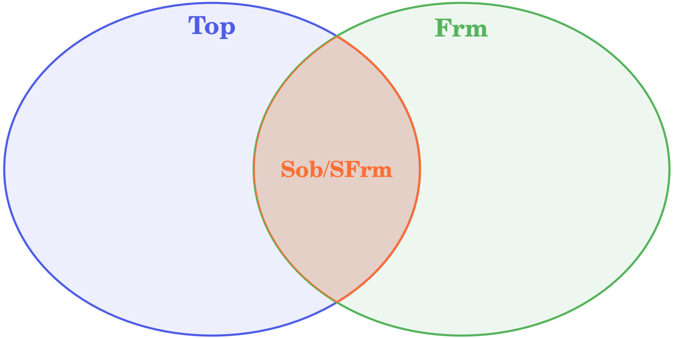
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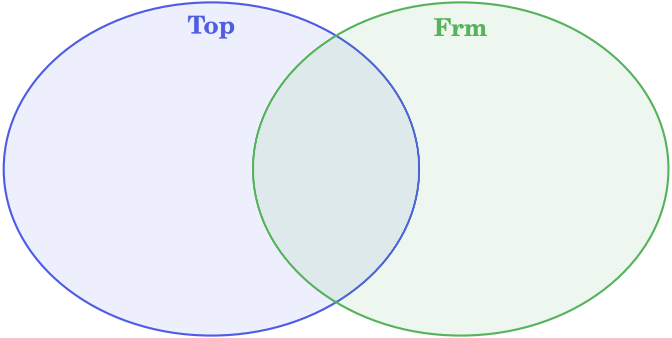
Sob/SFrm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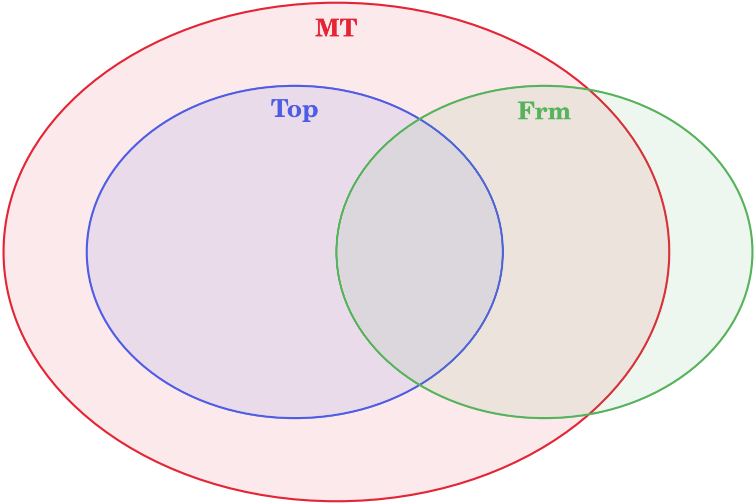
MT
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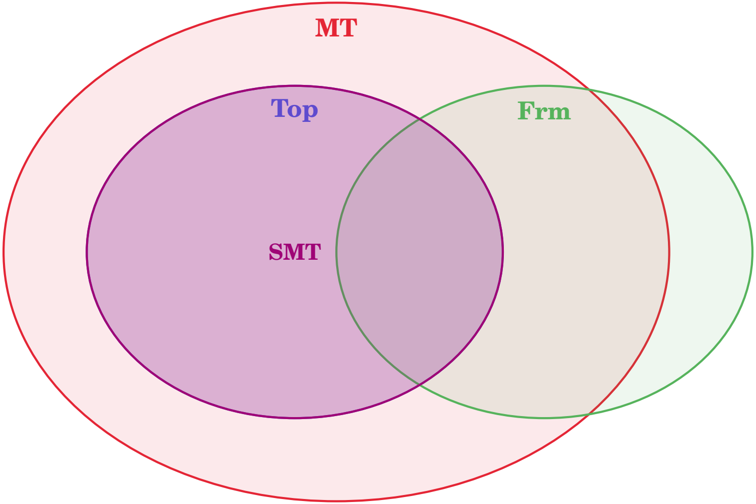
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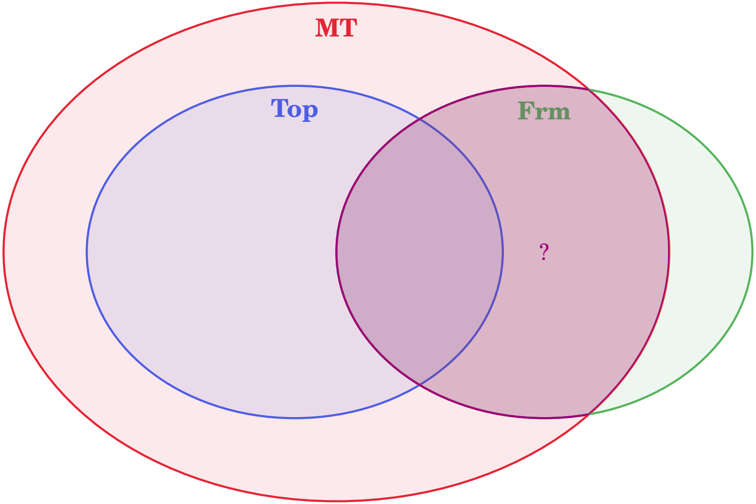
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# Open elements of an MT-algebra

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For instance, $\mathcal{O}(N(L)_{\neg\neg}) \cong L$. Equivalently, the Funayama envelope $\mathcal{F}(L)$ of a frame $L$.

# The Funayama envelope

A subset $S$ of a complete lattice $L$ has an **exact** join provided $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$ for all $a \in L$. Thus, a frame is a complete lattice where every join is exact.

[7] N. Funayama. "Imbedding infinitely distributive lattices completely isomorphically into Boolean algebras". In: *Nagoya Math. J.* 15 (1959), pp. 71–81

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For a frame $L$, the embedding $L \hookrightarrow \mathcal{F}(L)$ induces an interior operator, turning $\mathcal{F}(L)$ into an MT-algebra with $\mathcal{O}(\mathcal{F}(L)) \cong L$.

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# Local compactness in MT-algebras

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## Lemma ([BR25])

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**Lemma ([BR25])**

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(The converse holds for sober MT-algebras.)

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In a sense, the algebra is degenerate since its open part is trivial.

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Evidently, we want the MT-algebra to be determined in some sense by its interior, or equivalently, by its open elements.

For this we generalize separation axioms of topological spaces to MT-algebras.

Separation axioms weaker than or equal to $T_2$ are infamously difficult (or even impossible) to describe in the setting of frames.[9]

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The separation axioms $T_i$ for $i = 0, \frac{1}{2}, 1, 2, 3, 3\frac{1}{2}, 4$ were formulated in [BR23]. These formulations are faithful: a space $X$ is $T_i$ iff $\mathscr{P}(X)$ is a $T_i$-algebra.

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These definitions are also compatible with frame theory: under mild assumptions (e.g., $T_1$), $M$ is $T_i$ iff $\mathcal{O}(M)$ is $T_i$ for $i = 3, 3\frac{1}{2}, 4$.

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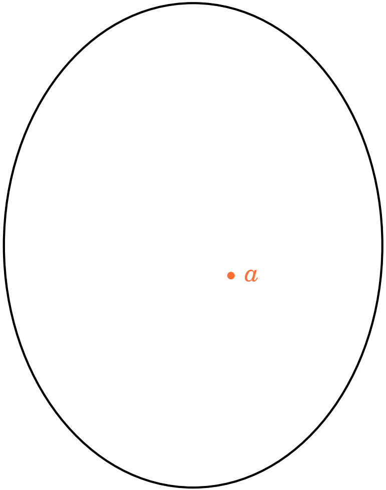

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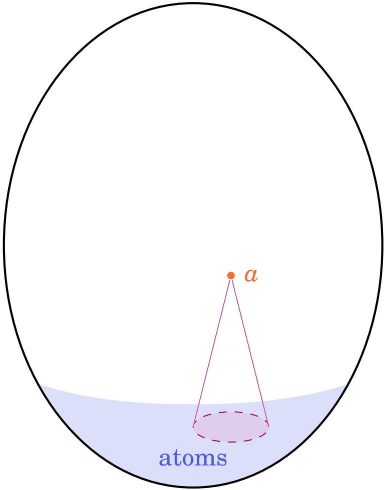

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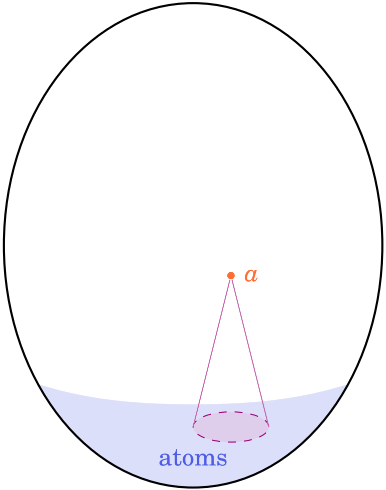

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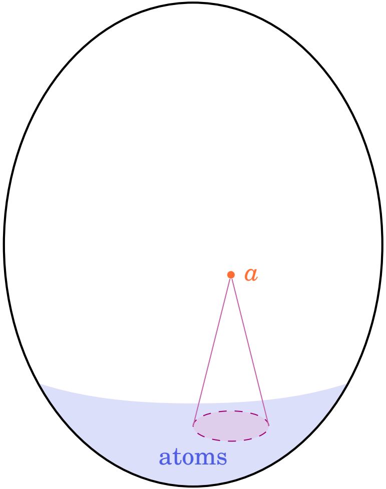

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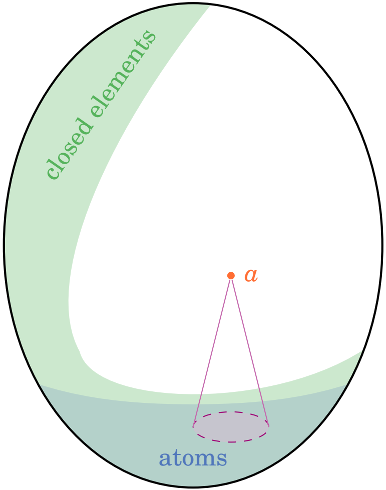

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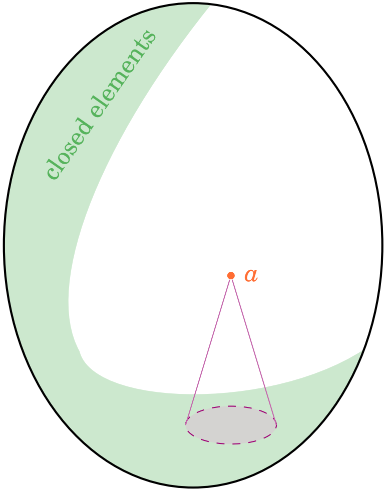

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- $X$ is $T_2$ iff $\{x\} = \bigcap\{\overline{U} \mid x \in U \in \Omega(X)\}$ for all $x \in X$.

These conditions give the intuition for the separation in MT-algebras.

# $T_0$, $T_D$ and $T_1$ in MT-algebras

Let $M$ be an MT-algebra and $a \in M$. We call $a$ **closed** if $a = \Diamond a := \neg\Box\neg a$, and **saturated** if it is a meet of open elements.

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## Remark

The higher separation axioms $T_3$–$T_5$ are defined similar to how they are defined for frames.

# $T_D$ and the Funayama envelope

If $L$ is a frame, then $\mathscr{F}(L)$ is a $T_D$-algebra. In fact, an MT-algebra $M$ is $T_D$ iff $M \cong \mathscr{F}(\mathscr{O}(M))$.

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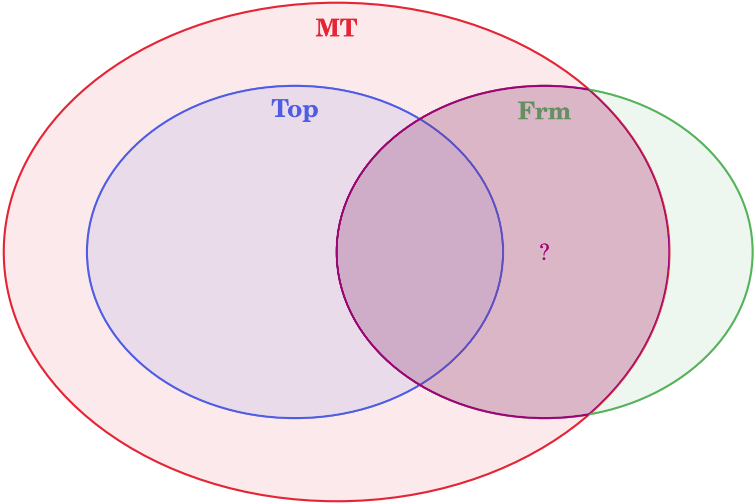
In this sense, MT-algebras are a generalization of both topological spaces and frames.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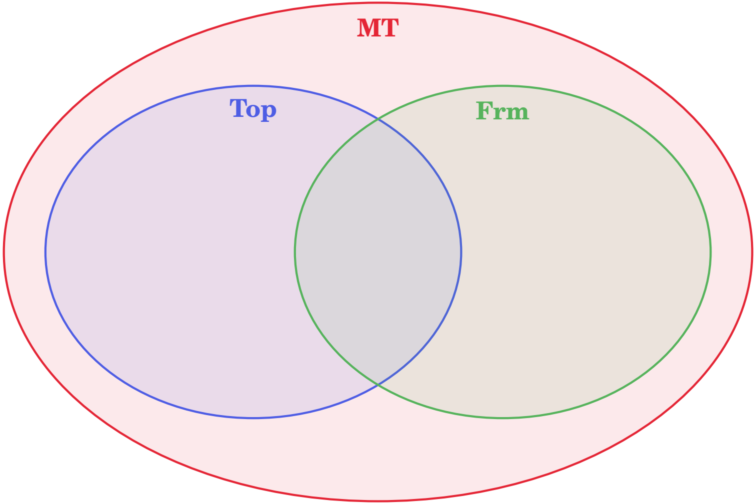
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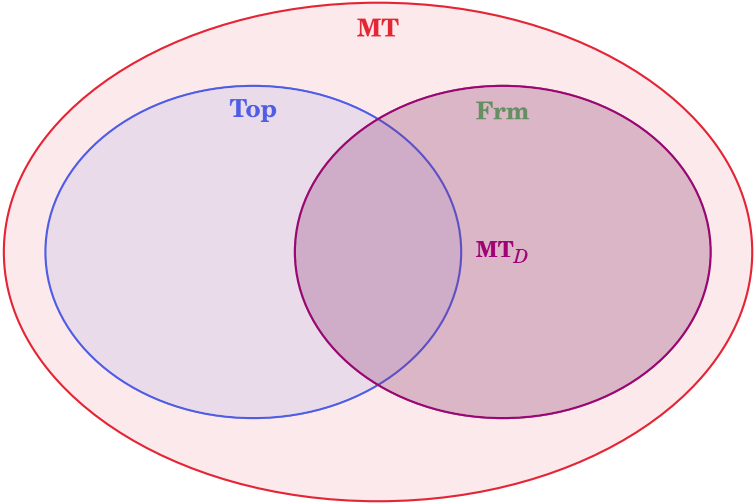
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# Sobriety in MT-algebras

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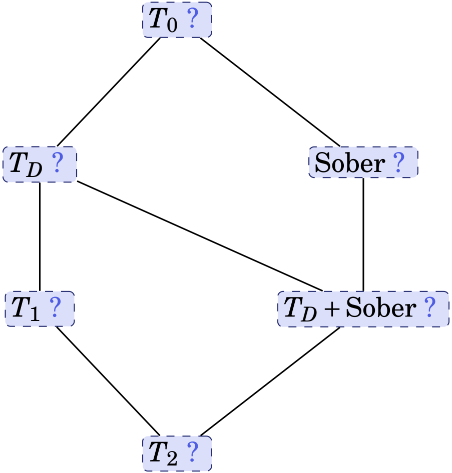
**Remark**

The Hofmann–Mislove Theorem generalizes to sober MT-algebras, see [BR25].

# 主要问题
# The main question

# Which locally compact MT-algebras are spatial?



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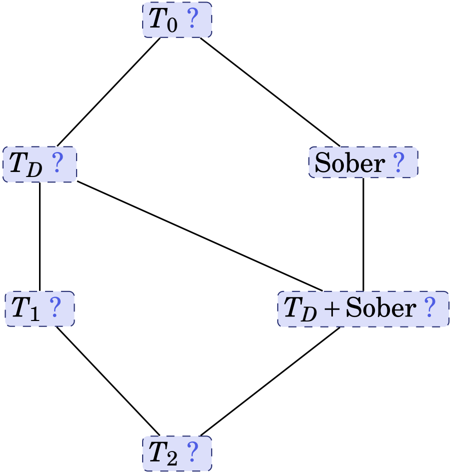

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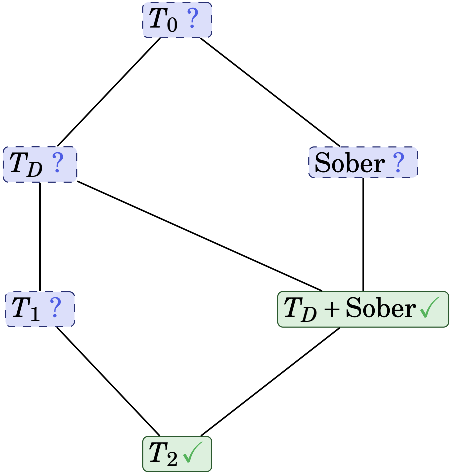

# The sober case and Raney extensions

The spatiality of locally compact sober MT-algebras was one of the main issues motivating this work.

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Roughly speaking, just as frames correspond to lattices of open sets, Raney extensions correspond to lattices of saturated sets.

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For any MT-algebra $M$, the lattice of saturated elements $\mathcal{S}at(M)$ forms a Raney extension. Conversely, for any Raney extension $C$, the Funayama envelope $\mathscr{F}(C)$ is a $T_0$-algebra.

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**Remark**

As with frames, this can be turned into a categorical equivalence.

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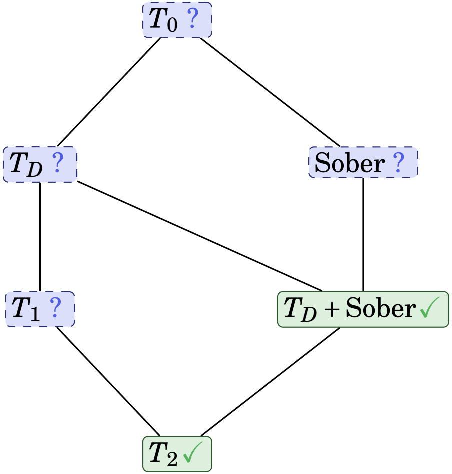
**Theorem ([BMRS26])**

*There exist locally compact sober MT-algebras that are not spatial.*

# Which locally compact MT-algebras are spatial?



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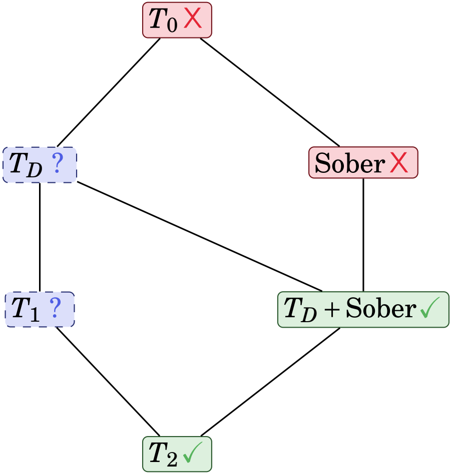

# The $T_D$ case

**Lemma ([BMRS26])**

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**Corollary ([BMRS26])**

*Locally compact $T_D$-algebras are spatial.*

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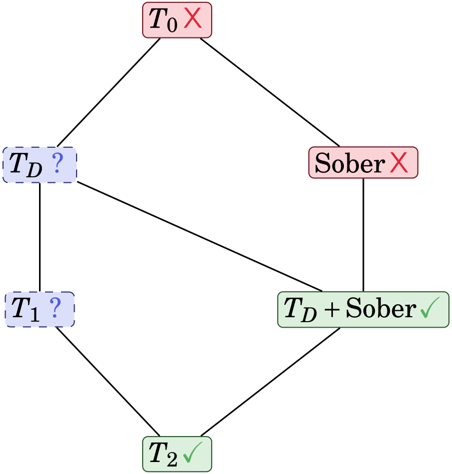

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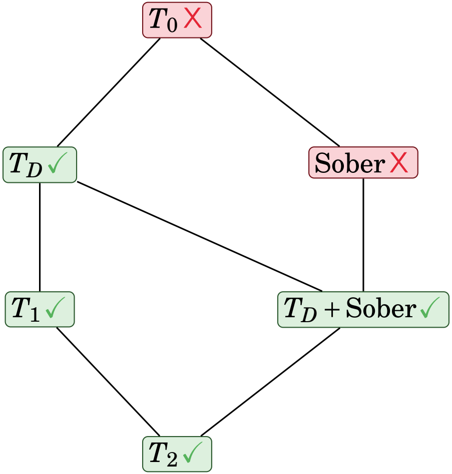

谢谢

Thank you

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# 附录
# Addendum

# Compactness and the axiom of choice

**Theorem ([BMRS26])**

*The following conditions are equivalent to the axiom of choice.*

1. *Every nontrivial compact MT-algebra contains a nonzero minimal closed element.*
2. *Every nontrivial compact $T_0$-algebra contains a closed atom.*
3. *Every nontrivial compact $T_D$-algebra contains a closed atom.*

**Question**

Is the condition that every nonempty compact $T_0$-space contains a closed singleton equivalent to the axiom of choice?

# The spatiality of continuous frames

**Theorem (see, e.g., [BMRS26])**

*Compact $T_1$-algebras are spatial.*

Isbell's Spatiality Theorem states that compact subfit frames are spatial. Subfit frames correspond precisely to $T_1$-algebras. Consequently, Isbell's Spatiality Theorem can be derived from the spatiality of compact $T_1$-algebras.

**Question**

Can we explain the spatiality of continuous frames via the spatiality of locally compact $T_D$-algebras?

The problem is that core compact $T_D$-algebras need not be locally compact.

# Normal MT-algebras

**Theorem ([BR23])**

*For all* $i \in \{0, D, 1, 2, 3, 3\frac{1}{2}, 5\}$, *if* $M$ *is a* $T_i$*-algebra, then* $\mathcal{at}(M)$ *is a* $T_i$*-space.*

A $T_1$-algebra $M$ is **normal** or a **$T_4$-algebra** if it for all closed $c, d$ such that $c \wedge d = 0$ there exist $u, v \in \mathcal{O}(M)$ such that $c \leq u$, $d \leq v$, and $u \wedge v = 0$.

**Question**

If $M$ is a $T_4$-algebra, is $\mathcal{at}(M)$ a $T_4$-space?

Presumably the answer is no since subspaces of normal spaces need not be normal.

# The functor $\mathfrak{O}$

The functor $\mathfrak{O} : \mathbf{MT} \to \mathbf{Frm}$ has no left or right adjoint.

**Question**

For which subcategories does the restriction of the functor have an adjoint?

# Pointfree dcpos

Posets correspond to $T_0$ Alexandroff spaces.

Call an MT-algebra, **Alexandroff** if every saturated element is open. Think of Alexandroff $T_0$-algebras as a pointfree version of posets.

**Question**

Describe when an Alexandroff $T_0$-algebra is a dcpo, and define the Scott topology on it. Provide an example of a pointless dcpo.