Math 1220G – Lecture Notes

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Warning

I provide these notes for the convenience of my students. Be aware that there might be mistakes and oversights in the text. When reading these notes you might first want to go over the material in the appendix.

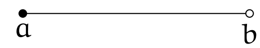
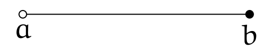
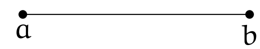
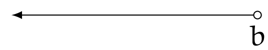
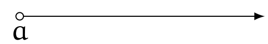
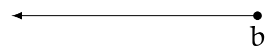
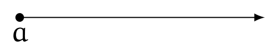
If you find any typos/errors please let me now.

Contents

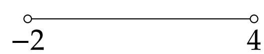
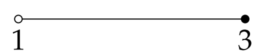
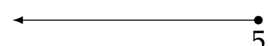
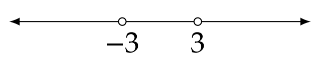
1.1 Sets of Real Numbers and the Cartesian Plane

Definition 1.1.1. An *interval* is the collection of all real numbers between two given real numbers (including or excluding those particular numbers).

Notation 1.1.2. Let $a, b \in \mathbb{R}$ and $a < b$. There are three ways of writing an interval.

Set of real numbers	Interval notation	Real line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leqslant x < b\}$	$[a, b)$	
$\{x \mid a < x \leqslant b\}$	$(a, b]$	
$\{x \mid a \leqslant x \leqslant b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid a < x\}$	(a, ∞)	
$\{x \mid x \leqslant b\}$	$(-\infty, b]$	
$\{x \mid a \leqslant x\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

Example 1.1.3. Intervals can be used in many ways to define segments of the real line.

Set of real numbers	Interval notation	Real line
$\{x \mid -2 < x < 4\}$	$(-2, 4)$	
$\{x \mid 1 \leqslant x < 3\}$	$(1, 3]$	
$\{x \mid x \leqslant 5\}$	$(-\infty, 5]$	
$\{x \mid x \leqslant -2 \text{ or } x \geqslant 2\}$	$(-\infty, -2] \cup [2, \infty)$	
$\{x \mid x \neq \pm 3\}$	$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$	

Exercise. Express the following sets using interval notation. Simplify if possible.

a) $\{x \mid x < 1\}$

b) $\{x \mid 0 \leqslant x < 1\}$

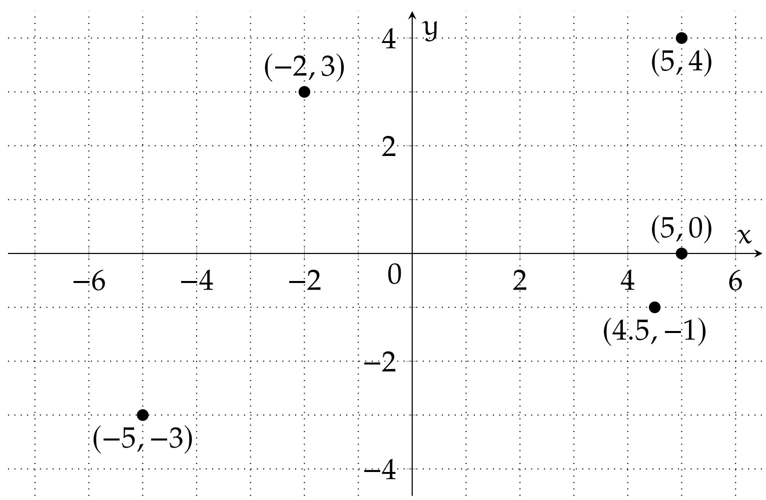
c) $\{x \mid x > 0\} \cup \{x \mid x < 0\}$

d) $\{x \mid x > -1\} \cap \{x \mid x < 1\}$

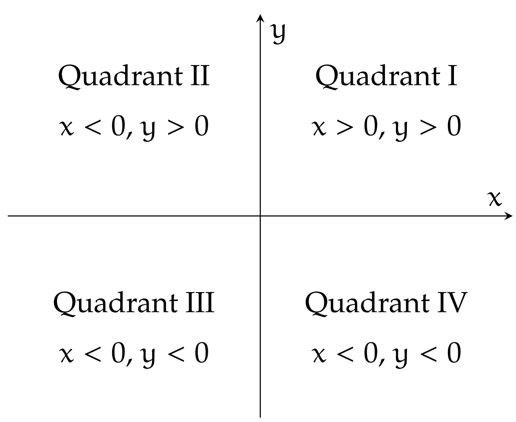
Historical Note. Next, we will introduce the Cartesian plane. The story goes that the French mathematican and philosopher René Descartes was lying in bed one night, watching a fly move across the ceiling. He realized that he could describe the fly's position by specifying how far it was from two perpendicular walls in the room. This insight led him to the idea that any point in space could be uniquely described by its distance along two perpendicular directions.

Definition 1.1.4. A *point* in the Cartesian plane is an ordered pair (x, y), where $x, y \in \mathbb{R}$.

Example 1.1.5. Let's plot some points in the Cartesian plane.

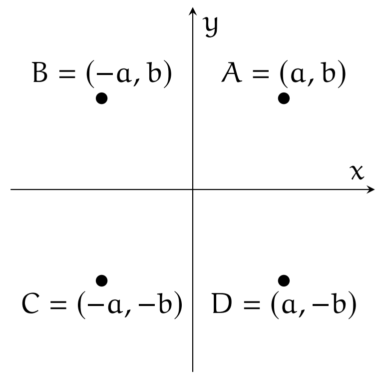


Remark 1.1.6. There are four different quadrants on the plane. They are labeled with Roman numerals and proceed counterclockwise around the plane:



Exercise. In which quadrant do the points of the previous example lie?

Remark 1.1.7. An important concept in the Cartesian plane is symmetry between points. Consider the following graph.



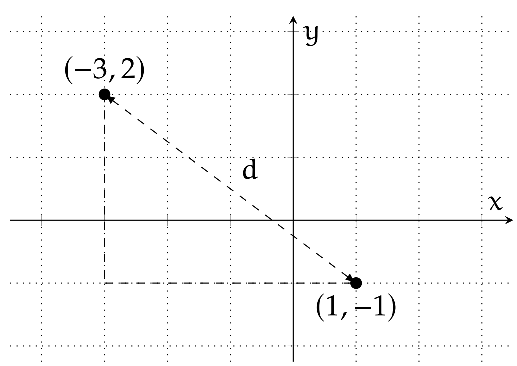
There are three types of symmetry and reflections.

(x**-axis**) A and D are <u>*symmetric about the x-axis*</u> since they have the same x-coordinate but opposite y-coordinates.

(y**-axis**) A and B are <u>*symmetric about the y-axis*</u> since they have the same y-coordinate but opposite x-coordinates.

(**origin**) A and C are <u>*symmetric about the origin*</u> since they have opposite x- and y-coordinates.

Example 1.1.8. Let's find the distance between the of points $(1, -1)$ and $(-3, 2)$. Consider the following graph:

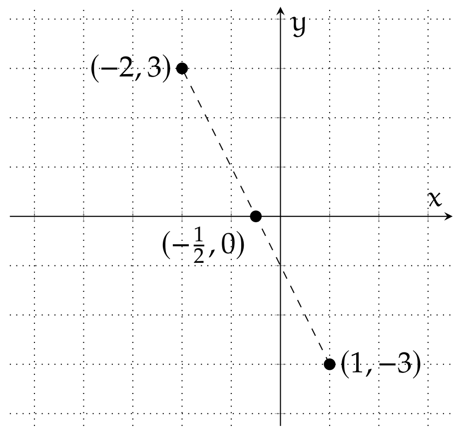


To find the distance we use the Pythagorean Theorem. Indeed it says that the distance d square is equal to horizontal difference squared and the vertical difference squared. Thus, the distance d between $(-2, 3)$ and $(1, -3)$ is

$$d = \sqrt{(-3-1)^2 + (5-2)^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Exercise. Find the distance between the points $(4, -4)$ and $(-2, 1)$.

Example 1.1.9. Last, let's find the point exactly between two points. In other words, the midpoint on the line segment connecting two points. Let's consider the points $(-2, 3)$ and $(1, -3)$. To find the midpoint we can simply take the average of the respective coordinates. That is the x-coordinate of the midpoint is $\dfrac{-2+1}{2} = -\dfrac{1}{2}$ and the y-coordinate is $\dfrac{3+-3}{2} = 0$.



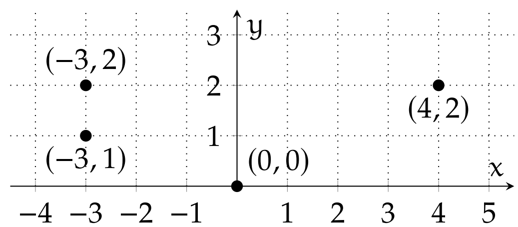
Exercise. Find the midpoint of the line segment connecting the points $(-2, 3)$ and $(-4, -1)$.

1.2 Relations

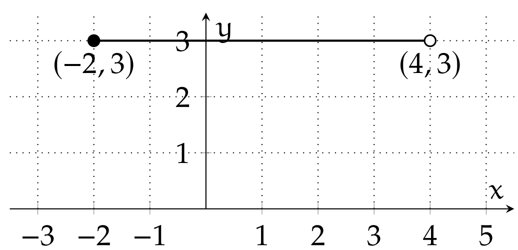
Definition 1.2.1. A *relation* is a set of points in the plane.

Remark 1.2.2. We have already seen how to graph points in the Cartesian plane. Naturally, to graph relations we simply graph all the points that are elements of the relation.

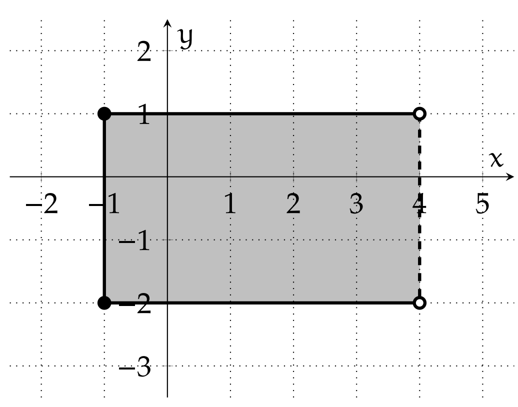
Example 1.2.3. For example, to graph $A = \{(0,0),(-3,1),(4,2),(-3,2)\}$ we just graph the four points belonging to A.



It is not always possible to graph every point individual. Consider $B = \{(x,3) \mid x \in [-2,4)\}$. There are infinitely many points in B, so it would take quite a while to graph all of them. Observe, that B contains all the points on the line connecting the points $(-2,3)$ and $(4,3)$. Therefore, we draw a line from $(-2,3)$ and $(4,3)$ to graph B, and to indicate that $(-2,3) \in B$ is included and $(4,3) \notin B$ we use $\bullet$ and $\circ$, at the respective end points of the line segment.



Last, we graph the relation $C = \{(x,y) \mid x \in [-1,4) \text{ and } y \in [-2,1]\}$. Observe that C describes a rectangle with corners at $(-1,-2)$, $(4,-2)$, $(-1,1)$, and $(4,1)$. We graph C by shading this rectangle. For the edges we draw the lines that are contained in C and uses dashes for those that are not.



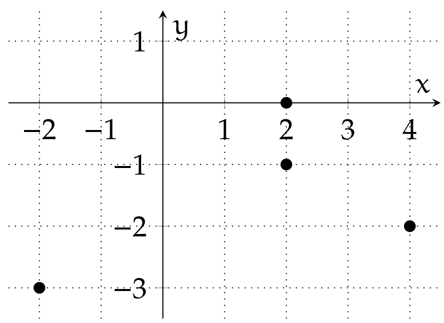
Exercise. Graph the following relations.

a) $\{(1,3),(1,1),(5,-2),(-1,2),(4,3)\}$

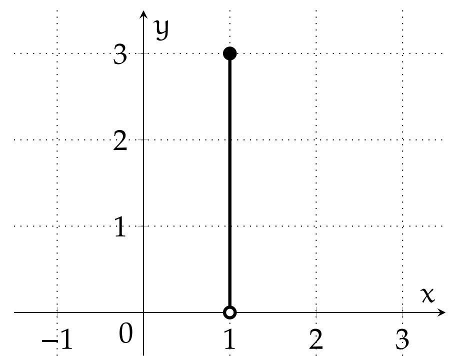
b) $\{(x,-4) \mid 0 < x \leqslant 3\}$

Exercise. Describe the relations corresponding to the following graphs using set notation.

a)



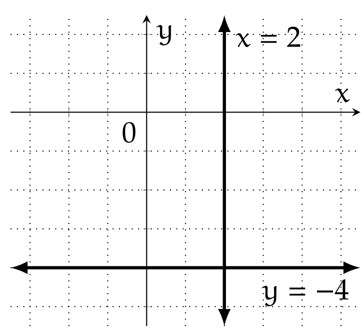
b)



Definition 1.2.4. Let $a \in \mathbb{R}$.

- The graph of the equation $x = a$ is a *vertical line* through $(a, 0)$.
- The graph of the equation $y = a$ is a *horizontal line* through $(0, a)$.

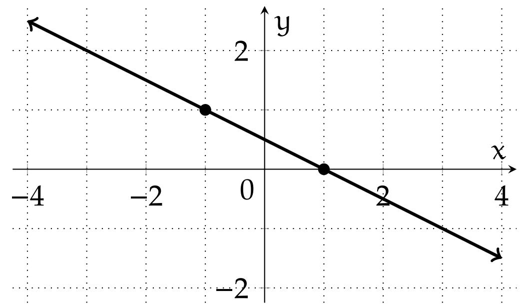
Example 1.2.5. Below is a graph of the vertical line $x = 2$ and the horizontal line $y = -4$



At what point do the lines intersect? What are the intercepts?

Definition 1.2.6. The *graph of an equation* is the graph of the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Example 1.2.7. To graph $x + 2y = 1$ we graph the relation $\{(x, y) \mid x + 2y = 1\}$. We will later see that $x + 2y = 1$ is a linear equation, in other words, it describes a line. Thus, to draw it we must simply find two points that satisfy the equation. To this, you could pick some random values for x and solve for y (or vice versa). For example, $(-1, 1)$ and $(1, 0)$ are points that satisfy the equation. We find the following graph.



Exercise. Graph the following equations.

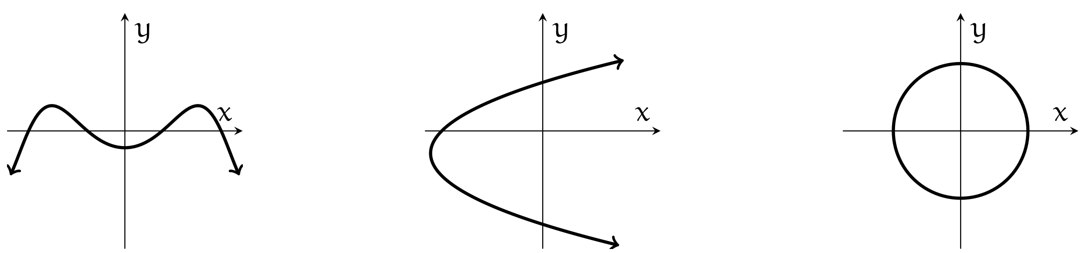
a) $x = 1$ b) $y = -1$ c) $x = y$

Definition 1.2.8. Suppose you are given some graph.

- A point on the graph which is also on the x-axis is called an *x-intercept* of the graph.
- A point on the graph which is also on the y-axis is called an *y-intercept* of the graph.

Note 1.2.9. Note that intercepts are points, so they have to be written as ordered pairs.

Exercise. Find all the x- and y-intercepts of the following graphs.



a) b) c)

Remark 1.2.10. Given an equation involving x and y, we can find the intercepts of the graph analytically as follows:

- x-intercepts have the form $(x, 0)$; set $y = 0$ in the equation and solve for x.
- y-intercepts have the form $(0, y)$; set $x = 0$ in the equation and solve for y.

Example 1.2.11.

- Consider the equation $3x - y = 9$. We will start by finding the y-intercept (which usually is the easiest one). We set $x = 0$ to get

$$3(0) - y = 9 \quad \Longrightarrow \quad -y = 9 \quad \Longrightarrow \quad y = -9$$

For the x-intercept, we set $y = 0$ to get

$$3x - 0 = 9 \quad \Longrightarrow \quad 3x = 9 \quad \Longrightarrow \quad x = 3$$

Hence, the only y-intercept is $(0, -9)$, and the only x-intercept is $(3, 0)$.

- Graphs can have multiple intercepts of each kind. Consider the equation $x^2 + y^2 = 1$. To find the y-intercepts, we set $x = 0$ to get

$$0^2 + y^2 = 1 \implies y^2 = 1 \implies y = \pm\sqrt{1} = \pm 1$$

Thus, the y-intercepts are $(0, -1)$ and $(0, 1)$.

To find the x-intercepts, we set $y = 0$ to get

$$x^2 + 0^2 = 1 \implies x^2 = 1 \implies x = \pm 1$$

Thus, the y-intercepts are $(0, -1)$ and $(0, 1)$, and the x-intercepts are $(-1, 0)$ and $(1, 0)$.

- Graphs of equations can have no intercepts. Consider the equation $xy = 1$. Plugging in $x = 0$ or $y = 0$ gives an unsolvable equation. Thus, the graph of this equation has no intercepts.

Exercise. Find the x- and y-intercept(s) of the following graphs, if any exist.

a) $y = 4x + 2$ b) $x + y = 10$ c) $x^2 - y = 1$ d) $y = \sqrt{x - 1}$

1.3 Introduction to functions

Definition 1.3.1. A *function* is a relation in which each x-coordinate is matched with only one y-coordinate.

Example 1.3.2.

- $\{(-2,1),(1,3),(1,4),(3,-1)\}$ is not a function since $(1,3)$ and $(1,4)$ are both in R. That means there are two y-coordinates for a single x-coordinate.
- $\{(-2,1),(1,3),(2,3),(3,-1)\}$ is a function. Every x-coordinate that appears in a pair in S matched with only one y-coordinate.

Exercise. Which of the following relations are functions?

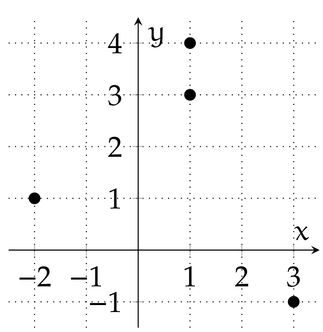
a) $\{(-3,4),(6,8),(6,6)\}$

b) $\{(7,9),(9,7),(6,-3)\}$

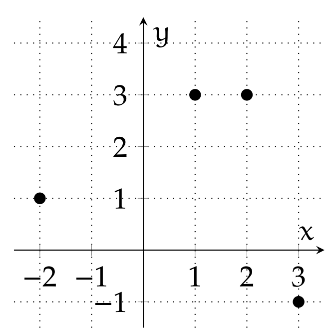
c) $\{(x,y) \mid y = x^2\}$

d) $\{(x,y) \mid x = y^2\}$

Remark 1.3.3. Consider the graphs of the relations from the previous example.



Graph of R.

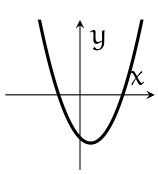


Graph of S.

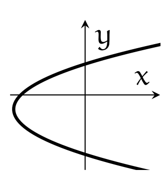
When two points of a relation have the same x-coordinate they lie on the same vertical line. Thus, we can see from the graph that R is not a function; it contains two points on the same vertical line. S does not have any points on the same vertical line.

Fact 1.3.4 (Vertical Line Test)**.** A relation is a function if and only if no two points on its graph lie on the same vertical line.

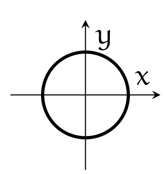
Example 1.3.5. Consider the following graphs of some relations.



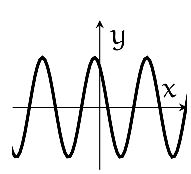
Graph A



Graph B



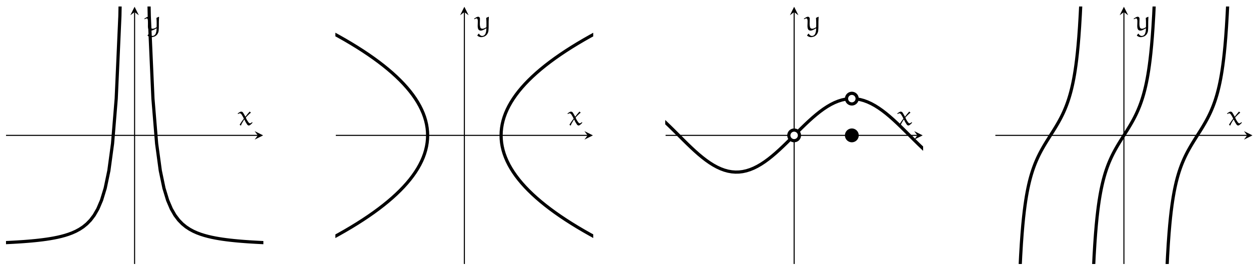
Graph C



Graph D

Using the vertical line test we see that only Graphs A and D represent functions.

Exercise. Determine which of the following graphs are functions:



Definition 1.3.6. Suppose f is a function.

(1) The <u>*domain*</u> of f is the set $\text{dom}(f) := \{x \mid (x, y) \in f\}$

(2) The <u>*range*</u> of f is the set $\text{ran}(f) := \{y \mid (x, y) \in f\}$

Remark 1.3.7. In other words, $\text{dom}(f)$ is the set of x-coordinates of points on the graph of f, and $\text{ran}(f)$ is the set of y-coordinates of points on the graph of f.

Example 1.3.8.

- Let $f = \{(0,3), (1,1), (2,4), (3,1), (4,5)\}$. Then $\text{dom}(f) = \{0,1,2,3,4\}$ and $\text{ran}(f) = \{3,1,4,5\}$.
- Let $g = \{(x,1) \mid x \in \mathbb{R}\}$. Then $\text{dom}(g) = (-\infty, \infty)$ and $\text{ran}(g) = \{1\}$.
- Let $h = \{(0,0), (0,2)\}$. Then $\text{dom}(h)$ and $\text{ran}(h)$ do not exist since h is not a function.

Exercise. Determine whether the following relations are functions. If they are find the domain and range.

a) $F = \{(-2,1), (-1,3), (0,1)\}$ b) $G = \{(42,1), (1,42), (42,42)\}$ c) $H = \{(x,y) \mid y = x + 1\}$

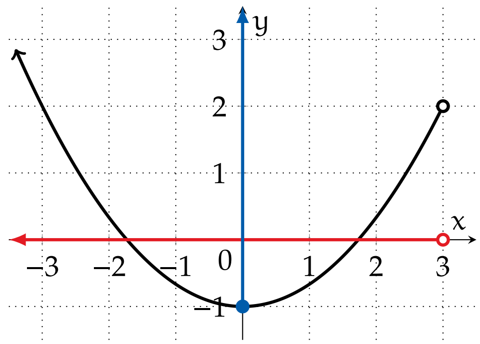
Remark 1.3.9. It is easy to find the domain and range of a function f given its graph.

(1) To find $\text{dom}(f)$, "project" the graph of f onto the x-axis. The set of values on the x-axis covered by the graph is $\text{dom}(f)$.

(2) To find $\text{ran}(f)$, "project" the graph of f onto the y-axis. The set of values on the y-axis covered by the graph is $\text{ran}(f)$.

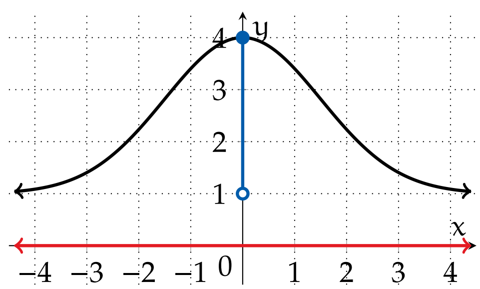
Example 1.3.10.

- Suppose the function f has the following graph.



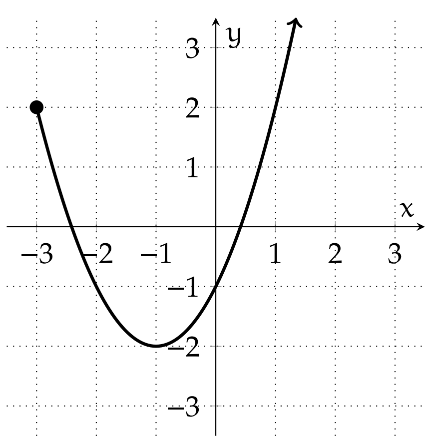
When we project the graph to the x- and y-axes we obtain the following red and blue graphs. We see $\operatorname{dom}(f) = (-\infty, 3)$ and $\operatorname{ran}(f) = [-1, \infty)$.

- Suppose function g has the following graph.

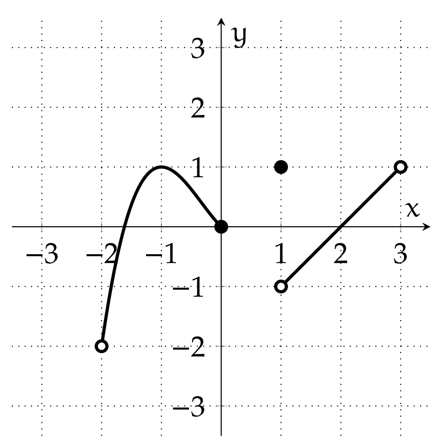


After projecting, we see that $\operatorname{dom}(g) = (-\infty, \infty)$ and $\operatorname{ran}(g) = (1, 4]$.

Exercise. Determine the domain and range of the following functions. Give your answer in interval notation.



Graph of f



Graph of g

1.4 Function notation

Remark 1.4.1. Functions usually describe processes. Mathematically, the process by which x-coordinates are matched with y-coordinates. In this sense, the domain is the set of inputs and the range is the the set of outputs. A function f is process which maps an input x to an output y.

Notation 1.4.2. Since the output of a function f is completely determined by the input x, we symbolize the output with function notation: $f(x)$, read "f of x."

In other words, $f(x)$ is the output which results by applying the process f to the input x. That is, $f(x) = y$ if and only if the pair (x, y) is in the relation that represents f.

Since the value of y is completely dependent on the choice of x, we usually call x the *argument* (or *independent variable*) of f, whereas y is often called the *dependent variable*.

Remark 1.4.3 (Common mistake)**.** Given a function f, the the parentheses in $f(x)$ do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully!

Example 1.4.4 (Function recipes)**.** The process of a function f is usually described using an algebraic formula. Consider the following examples.

- Suppose a function f takes a real number and performs the following two steps, in sequence

  i. add 3 ii. multiply by 5

  If we choose 4 as our input, in step i we add 3 to get $4+3 = 7$. In step ii, we multiply our result from step i by 5, which yields $7 \cdot 5 = 35$. Using function notation, we would write $f(4) = 35$ In general, if we use x for the input, applying the steps as before we find the term $5(x + 3)$. Hence for an input x, we get the output $f(x) = 5(x + 3)$.

- Suppose a function g is described by applying the following steps, in sequence

  i. multiply by 2 ii. add 3 iii. divide by 4

  We start with x and apply the steps in order. Step i gives us $x \cdot 2 = 2x$. Step ii gives us $2x + 3$. Step iii gives us $\frac{2x + 3}{4}$. Hence $g(x) = \frac{2x + 3}{4}$.

Example 1.4.5. Conversely, given algebraic formula we can describe a process (function recipe). For example, let $f(x) = (3(x - 1) + 4)^2$. To understand what this function do we read the function from inside out, i.e., look at the variable and see what mathematical operations are applied to it first. We find the following steps:

i. subtract 1 ii. multiply by 3 iii. add 4 iv. square it

Exercise. For each of the following, find an expression for a function that performs the following actions in order.

a) i: multiply by 2.
ii: add 3.
iii: take the square root.

b) i: add 3.
ii: multiply by 2.
iii: take the square root.

c) i: add 3.
ii: take the square root.
iii: multiply by 2.

Remark 1.4.6. Note that it is not always possible to find a nice function recipe. Consider the function $f(x) = (x-2)(x-3)$. Since there are instance of the variable x, we can't really say what step is done first. In a sense, we compute $x-2$ and $x-3$ in parallel and then multiply them together.

Example 1.4.7. Let $f(x) = x^2 - 3x - 4$. There are many calculations we can do with functions.

- $$\begin{aligned} f(0) &= 0^2 - 3(0) - 4 \\ &= -4 \end{aligned}$$
- $$\begin{aligned} f(-4) &= (-4)^2 - 3(-4) - 4 \\ &= 16 + 12 - 4 \\ &= 24 \end{aligned}$$
- $$f(\star) = \star^2 - 3\star - 4$$
- $$\begin{aligned} f(-x) &= (-x)^2 - 3(-x) - 4 \\ &= x^2 + 3x - 4 \end{aligned}$$
- $$\begin{aligned} f(3x) &= (3x)^2 - 3(3x) - 4 \\ &= 9x^2 - 9x - 4 \end{aligned}$$
- $$\begin{aligned} 3f(x) &= 3(x^2 - 3x - 4) \\ &= 3x^2 - 9x - 12 \end{aligned}$$
- $$\begin{aligned} f(x+3) &= (x+3)^2 - 3(x+3) - 4 \\ &= x^2 + 6x + 9 - 3x - 9 - 4 \\ &= x^2 + 3x - 4 \end{aligned}$$
- $$\begin{aligned} f(x) + f(3) &= x^2 - 3x - 4 + 3^2 - 3(3) - 4 \\ &= x^2 - 3x - 4 + 9 - 9 - 4 \\ &= x^2 - 3x - 8 \end{aligned}$$
- $$\begin{aligned} f\left(\frac{x}{3}\right) &= \left(\frac{x}{3}\right)^2 - 3\left(\frac{x}{3}\right) - 4 \\ &= \frac{x^2}{9} - x - 4 \end{aligned}$$
- $$\begin{aligned} \frac{f(x)}{3} &= \frac{x^2 - 3x - 4}{3} \\ &= \frac{1}{3}x^2 - x - \frac{4}{3} \end{aligned}$$

Observe the differences between d) and e), f) and g), and h) and i). In no way does function notation commute with basic operations!

Exercise. Let $f(x) = 2x - 12$. Evaluate and simplify the following expressions.

a) $f(1)$

b) $f(x+1)$

c) $f(\spadesuit)$

Remark 1.4.8. Recall, the domain of a function is the set of x-coordinates in the corresponding relation. If we define a a function with an algebraic expression it is not specified what the domain is. Then how do we know what the set of inputs of the function is? We assume implicitly that the domain is set of all real numbers such that the algebraic expression is well defined.

Definition 1.4.9. Let $f(x) =$ "expression using x" be a function. The *implied domain* of f is the set of real numbers for which "expression using x" is defined.

Example 1.4.10.

- Let $f(x) = \frac{1}{x}$. Evaluating $f(0)$ gives us a problem, because $f(0) = \frac{1}{0}$, but division by 0 is not defined. Division is defined for all other values, so the domain of f is all the real numbers but 0, i.e., $\text{dom}(f) = (-\infty, 0) \cup (0, \infty)$.
- Let $h(x) = \frac{2x}{x+1}$. Again, we have division in our function, and since we can't divide by 0 we must make sure that the denominator $x + 1 \neq 0 \implies x \neq -1$. Hence, $\text{dom}(h) = (-\infty, -1) \cup (-1, \infty)$, that is, everything but -1.
- Let $g(x) = \sqrt{x}$. Observe, $f(-2) = \sqrt{-2} = ?$. We can't take the square root of negative numbers. In other words, $\text{dom}(g) = [0, \infty)$.
- Finding the implied domain can become quite complicated, consider $i(x) = \frac{1}{1 - \sqrt{x+2}}$. For the right hand side to be defined, we need $1 - \sqrt{x+2} \neq 0$ and $x + 2 \geqslant 0$. Observe,

  $$1 - \sqrt{x+2} \neq 0 \implies 1 \neq \sqrt{x+2} \implies 1 \neq x + 2 \implies x \neq -1$$

  and

  $$x + 2 \geqslant 0 \implies x \geqslant -2.$$

  In interval notation, we need x to be in $(-\infty, -1) \cup (1, \infty)$ and in $[-2, \infty)$. Intersecting the two gives us the domain: $\text{dom}(i) = [-2, -1) \cup (-1, \infty)$.

Exercise. Find the domain of the following functions.

a) $f(x) = \sqrt{4x - 8}$ b) $g(x) = x^2 + 1$ c) $h(x) = \frac{4}{6 - \sqrt{x+3}}$ d) $r(x) = \frac{x^2}{x}$

Remark 1.4.11. We should have found $\text{dom}(r) = (-\infty, 0) \cup (0, \infty)$ in the previous exercise. If we had simplified $r(x) = \frac{x^2}{x} = x$, the we would have (erroneuously) found $\text{dom}(r) = (-\infty, \infty)$. It is crucial to find the implied domain before simplifying the function.

Notation 1.4.12 (Piecewise-defined functions)**.** A *piecewise-defined* function or *piecewise* function is a function in which more than one expression is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this

$$f(x) = \begin{cases} \text{expression 1} & \text{if } x \text{ is in domain 1} \\ \text{expression 2} & \text{if } x \text{ is in domain 2} \\ \text{expression 3} & \text{if } x \text{ is in domain 3} \end{cases}$$

The different lines in the equations are called *cases*.

Example 1.4.13.

- Let $f(x) = \begin{cases} 7x + 3 & \text{if } x \in (-\infty, 0) \\ 4 - 2x & \text{if } x \in (0, \infty) \end{cases}$

  Then

  - $f(-1) = 7(-1) + 3 = -7 + 3 = -4$ since $-4 \in (-\infty, 0)$ falls into the first case.
  - $f(3) = 4 - 2(3) = 4 - 6 = -2$ since $-3 \in (0, \infty)$ falls into the second case.
  - $f(0)$ is not defined because it falls in neither case.
  - $\mathrm{dom}(f) = (-\infty, 0) \cup (0, \infty)$ the union of both cases.

- Usually, in piecewise functions we do not use interval notation, e.g.

$$g(x) = \begin{cases} 2 - x & \text{if } x \leqslant 2 \\ \frac{1}{x-2} & \text{if } 2 < x < 4 \\ x - 4 & \text{if } x \geqslant 4 \end{cases}$$

  Then

  - $g(1) = 2 - 1 = 1$ since $1 \leqslant 2$ (first case).
  - $g(2) = 2 - 2 = 0$ since $2 \leqslant 2$ (first case).
  - $g(3) = \dfrac{1}{2-3} = \dfrac{1}{-1} = -1$ since $2 < 3 < 4$ (second case).
  - $g(4) = x - 4 = 0$ since $4 \geqslant 4$ (third case).
  - To find the dom(g), we need to write write the cases in interval notation and take the union:

$$\mathrm{dom}(g) = (-\infty, 2] \cup (2, 4) \cup [4, \infty) = (\infty, \infty).$$

- Note that expressions on the left do not always need to have a variable in them, e.g.

$$h(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$$

  Then

  - $h(-4) = 0$ since $-4 \leqslant 0$
  - $h(7) = 2(7) + 1 = 15$ since $7 > 0$

Exercise. Let $f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ 42 & \text{if } x \geqslant 2. \end{cases}$ Evaluate and simplify the following expressions.

a) $f(1)$ b) $f(2)$ c) $f(3)$

1.5 Function Arithmetic

Definition 1.5.1. Suppose f and g are functions and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. We define the following functions:

- The *sum* of f and g, denoted by $f + g$, is defined by the formula $(f + g)(x) = f(x) + g(x)$.
- The *difference* of f and g, denoted by $f - g$, is defined by $(f - g)(x) = f(x) - g(x)$.
- The *product* of f and g, denoted by fg, is defined by $(fg)(x) = f(x)g(x)$.
- The *quotient* of f and g, denoted by $\frac{f}{g}$, is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

Example 1.5.2. Let $f(x) = x^2 + 7$ and $g(x) = 3x - 1$. Then

$$\begin{aligned}(f + g)(0) &= f(0) + g(0)\\ &= (0^2 + 7) + (3(0) - 1)\\ &= (0 + 7) + (0 - 1)\\ &= 7 + (-1)\\ &= 7 - 1\\ &= 6\end{aligned}$$

$$\begin{aligned}(f - g)(1) &= f(1) - g(1)\\ &= (1^2 + 7) - (3(1) - 1)\\ &= (1 + 7) + (3 - 1)\\ &= 8 - 2\\ &= 6\end{aligned}$$

$$\begin{aligned}(fg)(-1) &= f(-1)g(-1)\\ &= ((-1)^2 + 7)(3(-1) - 1)\\ &= (1 + 7)(-3 - 1)\\ &= 8(-4)\\ &= -32\end{aligned}$$

$$\begin{aligned}\left(\frac{f}{g}\right)(2) &= \frac{f(2)}{g(2)}\\ &= \frac{2^2 + 7}{3(2) - 1}\\ &= \frac{4 + 7}{6 - 1}\\ &= \frac{11}{5}\end{aligned}$$

Exercise. Consider the following functions.

$f(x) = 2x + 1$ $\qquad$ $g(x) = 4 - x$ $\qquad$ $h(x) = \sqrt{x + 6}$

Compute the following values, if they exist.

a) $(f + g)(3)$ $\qquad$ b) $(g - f)(2)$ $\qquad$ c) $(fh)(3)$ $\qquad$ d) $\left(\frac{h}{g}\right)(-2)$

Example 1.5.3. Let $f(x) = x^2 - 4$ and $g(x) = x + 2$. Then

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= (x^2 - 4) + (x + 2)\\ &= x^2 + x - 2\end{aligned}$$

$$\begin{aligned}(g - f)(x) &= g(x) - f(x)\\ &= (x + 2) - (x^2 - 4)\\ &= x + 2 - x^2 + 4\\ &= -x^2 + x + 6\end{aligned}$$

$$\begin{aligned}(gg)(x) &= g(x)g(x)\\ &= (x+2)(x+2)\\ &= x^2+2x+2x+4\\ &= x^2+4x+4\end{aligned}$$

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\\ &= \frac{x^2}{x+2}\\ &= \frac{(x+2)(x-2)}{x+2}\\ &= x-2\end{aligned}$$

Remark 1.5.4. The previous example might make you believe that $\left(\frac{f}{g}\right)(-2) = -2-2 = -4$. However, when we evaluate without this simplified expression we find:

$$\left(\frac{f}{g}\right)(-2) = \frac{f(-2)}{g(-2)} = \frac{(-2)^2-4}{-2+2} = \frac{0}{0} = ?$$

Recall that $\left(\frac{f}{g}\right)(x)$ is not defined for x such that $g(x)=0$. Thus, $-2 \notin \operatorname{dom}\left(\frac{f}{g}\right)$.

In fact, to find the domain of such functions we proceed as we find the implied domain for any function. However, we need to work with unsimplified functions.

Example 1.5.5. We saw $\left(\frac{f}{g}\right) = \frac{x^2}{x+2}$ in the previous example. Thus, $\operatorname{dom}\left(\frac{f}{g}\right)$ is all the x such that $x+2 \neq 0$. Solving that gives:

$$x+2 \neq 0 \iff x \neq -2.$$

Hence, $\operatorname{dom}\left(\frac{f}{g}\right) = (-\infty,-2) \cup (-2,\infty)$.

Exercise. Let $f(x) = \sqrt{3x-9}$ and $g(x) = 4-x$. Determine the implied domain of the following functions.

a) $(f-g)(x)$

b) $\left(\frac{f}{g}\right)(x)$

Fact 1.5.6. Suppose f and g are functions.

- $\operatorname{dom}(f+g) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$.
- $\operatorname{dom}(f-g) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$.
- $\operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$.
- $\operatorname{dom}\left(\frac{f}{g}\right) = \operatorname{dom}(f) \cap \operatorname{dom}(g) \cap \{x \mid g(x) \neq 0\}$.

Remark 1.5.7. By the previous fact, $\operatorname{dom}(f+g) = \operatorname{dom}(f-g) = \operatorname{dom}(fg)$, so finding one gives all the others.

Example 1.5.8. Suppose $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x}$. Then it is easy to find that $\operatorname{dom}(f) = (-\infty, 0) \cup (0, \infty)$ and $\operatorname{dom}(g) = [0, \infty)$. Intersecting gives $\operatorname{dom}(f) \cap \operatorname{dom}(g) = (0, \infty)$. Consequently,

$$\operatorname{dom}(f + g) = \operatorname{dom}(f - g) = \operatorname{dom}(fg) = (0, \infty).$$

1.6 Graphs of Functions

Definition 1.6.1. The <u>*graph of a function*</u> f is the graph of the equation $y = f(x)$.

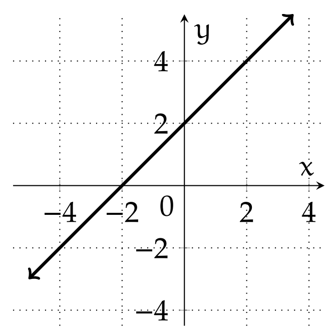
Remark 1.6.2. This is called the *Fundamental Graphing Principle for Functions*.

Fact 1.6.3. Let f be a function.

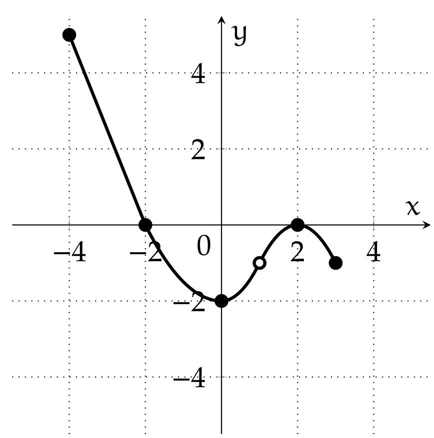
(1) The point (x, y) is on the graph of f if and only if $y = f(x)$.

(2) The graph of a function is exactly the graph of the relation underlying the function.

Example 1.6.4. Let $f(x) = x+2$. Then the graph of f is the graph of the set $\{(x, x+2) \mid x \in (-\infty, \infty)\}$:



Example 1.6.5 (♠). Consider the graph of $y = f(x)$ below.



We can find the following information by inspecting the graph:

- $f(0) = -2$, $f(-2) = 0$, $f(-4) = 5$, …
- $\operatorname{dom}(f) = [-4, 1) \cup (1, 3]$ and $\operatorname{ran}(f) = [-2, 5]$
- the x-intercepts are $(-2, 0)$ and $(2, 0)$
- the y-intercept is $(0, -2)$

Definition 1.6.6. Let f be a function. Then $z \in \text{dom}(f)$ is a *zero* of f if $f(z) = 0$.

Fact 1.6.7. The following are equivalent.

(1) z is a zero of f.

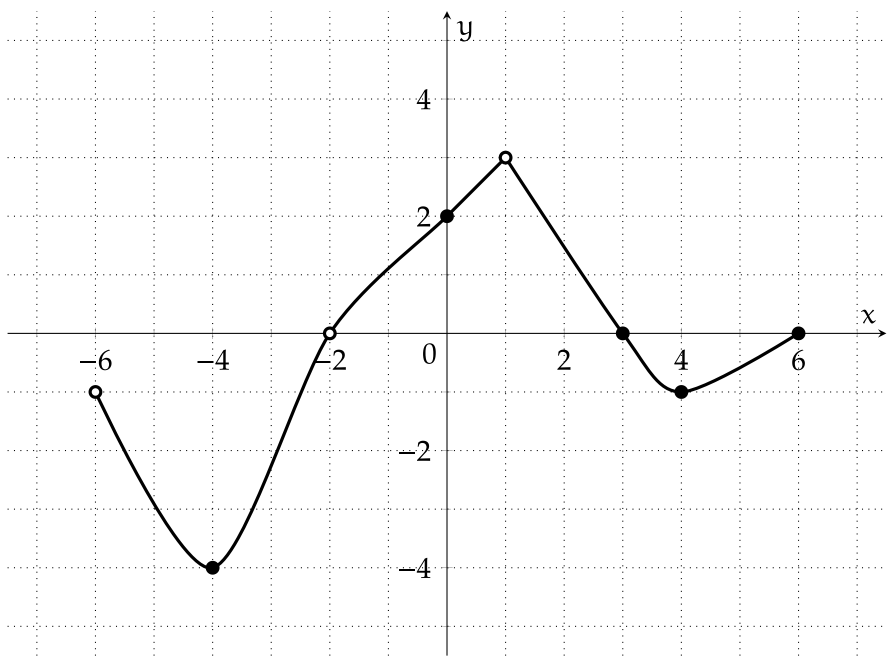
(2) $f(z) = 0$.

(3) $x = z$ is a solution to the equation $f(x) = 0$.

(4) $(z, 0)$ is an x-intercept of the graph of f.

Example 1.6.8 (♠)**.** The zeros of f are -2 and 2.

Exercise (◇)**.** Suppose g is a function with the following graph.



Find the following.

a) dom(g) b) ran(g) c) x-intercept(s) d) y-intercept e) zeros

Remark 1.6.9. Recall, a set of real numbers is an *interval* if it is a line segment. For example,

$(-3, 7)$ $(0, 42]$ $(-\infty, -5)$ $\{5\}$

are intervals, but

$(-2, 0) \cup (0, 5)$ $[0, 10] \cup \{15\}$ $(-\infty, -10) \cup (42, \infty)$ $\{0, 1\}$

are not intervals.

Definition 1.6.10. Suppose f is a function defined on an interval I. Then f is

- *increasing* on I if for all $a, b \in I$ we have that $a < b$ implies $f(a) < f(b)$.
- *decreasing* on I if for all $a, b \in I$ we have that $a < b$ implies $f(a) > f(b)$.
- *constant* on I if for all $a, b \in I$ we have that $f(a) = f(b)$.

We say f is *increasing* / *decreasing* / *constant* on an arbitrary set of real numbers if it is increasing / decreasing / constant on all intervals contained in that set.

Example 1.6.11 (♠).

- f is increasing on $[0, 1)$ and $(1, 2]$, i.e., on $[0, 1) \cup (1, 2]$.
- f is decreasing on $[-5, 0] \cup [2, 3]$.
- f is not constant on any (nontrivial) interval, that is, it is only constant on singletons.

Exercise (◇). Determine on which intervals g is

f) increasing g) decreasing h) constant

Definition 1.6.12. Suppose f is a function with $f(a) = b$. The value b is called the

- *maximum* of f if $b \geqslant f(x)$ for all $x \in \text{dom}(f)$.
- *minimum* of f if $b \leqslant f(x)$ for all $x \in \text{dom}(f)$.

Example 1.6.13.

- (♠). The maximum of f is 5 and the minimum of f is -2.
- Let $g(x) = x^2$. The minimum of g is 0 and g has no maximum.
- Let $h(x) = x$. h does neither have a maximum nor a minimum.

Remark 1.6.14. An interval is *open* if it can be written in interval notation using only round parentheses. That is, it can be written in one of the following forms: (a, b), (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$.

Definition 1.6.15. Suppose f is a function. Then a point $(a, f(a))$ on the graph of f is a

- *local maximum* of f if $\text{dom}(f)$ contains an open interval I such that $a \in I$ and for all $x \in I$ we have $f(a) > f(x)$.
- *local minimum* of f if $\text{dom}(f)$ contains an open interval I such that $a \in I$ and for all $x \in I$ we have $f(a) < f(x)$.

Example 1.6.16 (♠).

- f has a local maximum at $(2, 0)$ and a local minimum at $(0, -2)$.

- $(-4, 5)$ is <u>not</u> a local maximum of f since there is no open interval around -4 which is contained in the domain of f. Similarly, $(3, -1)$ is not a local minimum.

Exercise ($\diamond$)**.** Find the following of g, if they exist.

i) the maximum j) the minimum k) local maximum(s) l) local minimum(s)

Definition 1.6.17. Odd and even functions Let f be a function. Then f is

- *even* if $f(-x) = f(x)$ for all $x \in \text{dom}(f)$.
- *odd* if $f(-x) = -f(x)$ for all $x \in \text{dom}(f)$.

Example 1.6.18.

- Let $f(x) = 6x^5 + 3x^2 - 7$. Then

$$\begin{aligned} f(x) &= 6x^5 + 3x^2 - 7 \\ f(-x) &= 6(-x)^5 + 3(-x)^2 - 7 = -6x^5 + 3x^2 - 7 \\ -f(x) &= -(6x^5 + 3x^2 - 7) = -6x^5 - 3x^2 + 7 \end{aligned}$$

  None of $f(x)$, $f(-x)$, $-f(x)$ are equal, so f is neither even nor odd.

- Let $g(x) = \dfrac{4x^3}{10x^2 - x^4}$. Then

$$\begin{aligned} g(x) &= \frac{4x^3}{10x^2 - x^4} \\ g(-x) &= \frac{4(-x)^3}{10(-x)^2 - (-x)^4} = \frac{-4x^3}{10x^2 - x^4} \\ -g(x) &= -\frac{4x^3}{10x^2 - x^4} = \frac{-4x^3}{10x^2 - x^4} \end{aligned}$$

  We have $g(-x) = -g(x)$, so g is odd.

- Let $h(x) = \sqrt{x^8 + 3}$. Then

$$\begin{aligned} h(x) &= \sqrt{x^8 + 3} \\ h(-x) &= \sqrt{(-x)^8 + 3} = \sqrt{x^8 + 3} \\ -h(x) &= -\sqrt{x^8 + 3} \end{aligned}$$

  We have $h(x) = -h(x)$, so h is even

Remark 1.6.19.

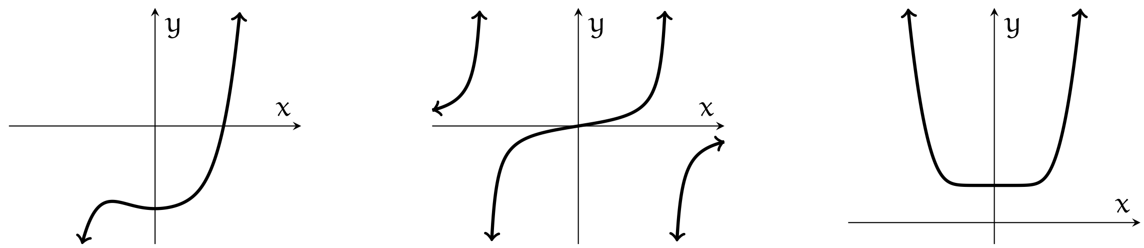
(1) We will see later why such functions are called respectively even and odd.

(2) Oddness and evenness of a function f corresponds to symmetries of the graph of f, see the following fact.

Fact 1.6.20. Let f be a function.

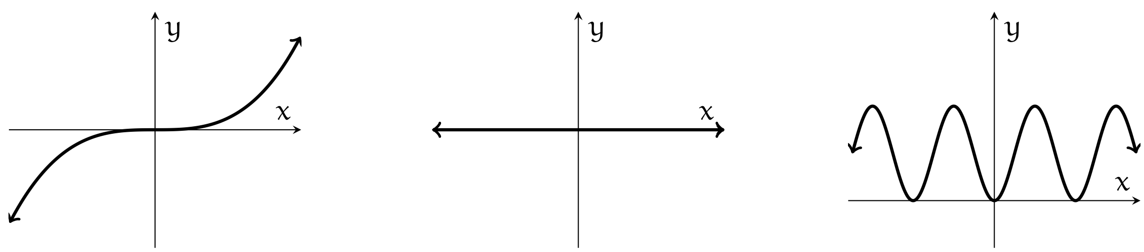
- f is even if and only if the graph of f is symmetric about the y-axis.
- f is odd if and only if the graph of f is symmetric about the origin.

Example 1.6.21. Consider the graphs of the functions of the previous example.



- The first graph has no symmetries, so the function is neither odd or nor even.
- The second graph is symmetric about the origin, hence the function is odd.
- The third graph is symmetric about the y-axis, hence the function is even.

Exercise. Determine whether the following functions are even, odd, both, or neither.



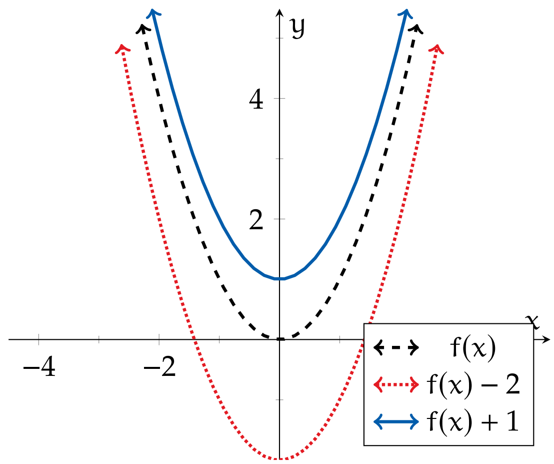
Remark 1.6.22. We have names for functions with graphs that are symmetric about the y-axis, and functions with graphs that are symmetric about the origin. We do not have a name for functions with a graph that is symmetric about the x-axis because such functions are not interesting. Namely, if f is symmetric about the x-axis, then $f(x) = -f(x)$ for all $x \in \text{dom}(f)$. Solving this equality, we get $f(x) = 0$. Hence, only trivial functions satisfy this symmetry.

1.7 Transformations

Fact 1.7.1 (Vertical shifts)**.** Let f be a function and k a positive number.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ up k units.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ down k units.

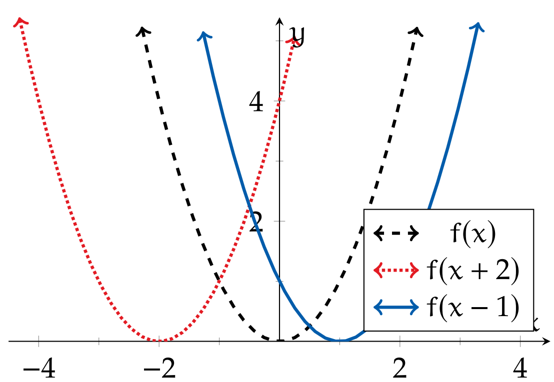
Example 1.7.2. Let $f(x) = x^2$. Then $y = f(x) + 1$ and $y = f(x) - 2$ are plotted as follows.



Fact 1.7.3 (Horizontal shifts)**.** Let f be a function and k a positive number.

- To graph $y = f(x + k)$, shift the graph of $y = f(x)$ to the left k units.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ to the right k units.

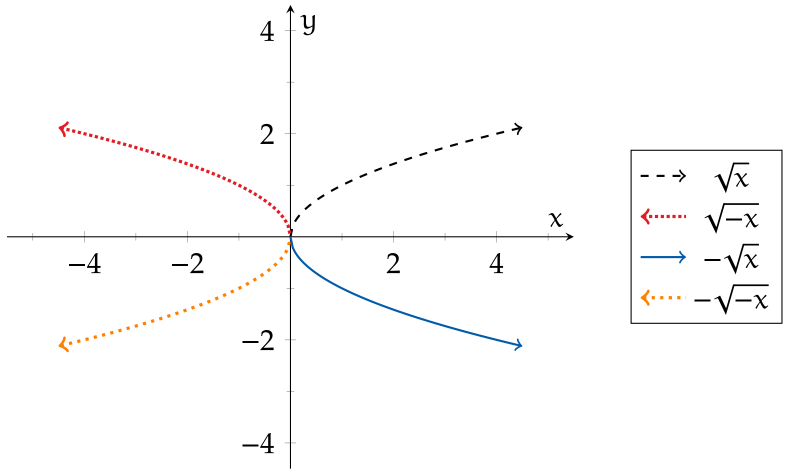
Example 1.7.4. Let $f(x) = x^2$. Then $f(x + 2)$ and $f(x - 1)$ are plotted below.



Fact 1.7.5 (Reflections)**.** Let f be a function.

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.
- To graph $y = -f(-x)$, reflect the graph of $y = f(x)$ about the origin.

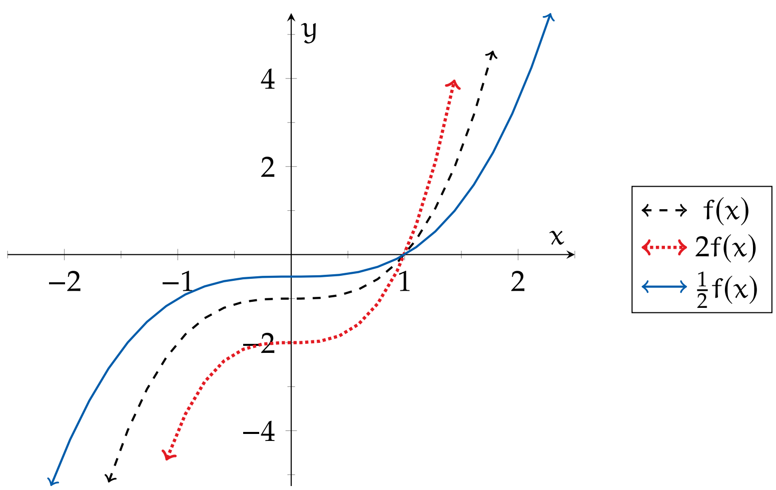
Example 1.7.6. Let $f(x) = \sqrt{x}$. Then $f(-x)$, $-f(x)$, and $f-(-x)$ are plotted below.



Fact 1.7.7 (Vertical scaling)**.** Let f be a function and k a positive number.

- To graph $y = kf(x)$, multiply all of the y-coordinates of points on the graph of f by k.
  - If $k > 1$, we say that the graph of f has undergone *vertical stretching* (or vertical expansion or vertical dilation).
  - If $k < 1$, we say that the graph of f has undergone *vertical shrinking* (or vertical compression or vertical contraction).

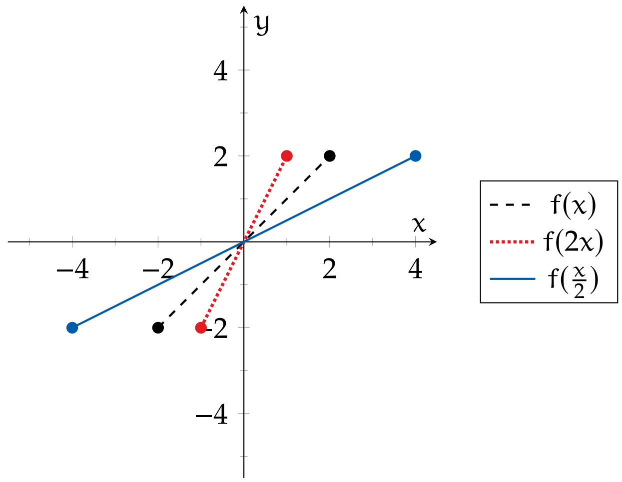
Example 1.7.8. Let $f(x) = x^3 + -1$. Then $y = 2f(x)$ and $y = \frac{1}{2}f(x)$ are plotted below.



Fact 1.7.9 (Horizontal scalings)**.** Let f be a function and k a positive number.

- To graph $y = f(kx)$, divide all of the x-coordinates of points on the graph of f by k.
  - If $k < 1$, we say that the graph of f has undergone *horizontal stretching* (or horizontal expansion or horizontal dilation).
  - If $k > 1$, we say that the graph of f has undergone *horizontal shrinking* (or horizontal compression or horizontal contraction).

Example 1.7.10. Let $f(x) = x$ with $\text{dom}(f) = [-2, 2]$. Then $y = f(2x)$ and $y = f(\frac{x}{2})$ are plotted below.



Fact 1.7.11. Let $f(x)$ be a function and $a, b, h, k \in (-\infty, \infty)$. If $g(x) = af(bx + h) + k$ then to graph g, sequentially do the following to the graph of f:

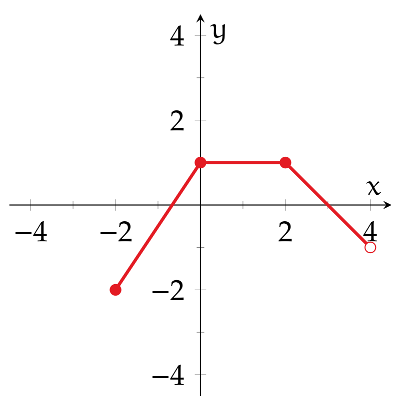
1. Horizontally shift by h. It is a left shift if $h > 0$, and a right shift if $h < 0$.

2. Horizontally scale by b. This is reflection about the y-axis if $b < 0$.

3. Vertically scale by a. This reflection about the x-axis if $a < 0$.

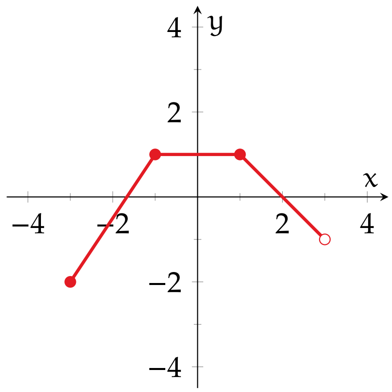
4. Vertically shift by k. This is an up shift if $k > 0$, and a down shift if $k < 0$.

Example 1.7.12. Consider the graph of the function $f(x)$ below.



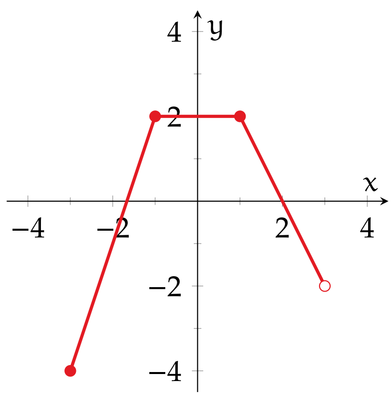
We find the graph $y = 2f(x+1) + 2$ in three steps.

1. Shift left by 1



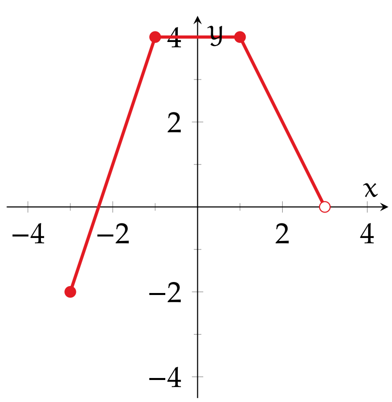
Graph of $f(x+1)$.

2. Scale vertically by 2



Graph of $2f(x+1)$.

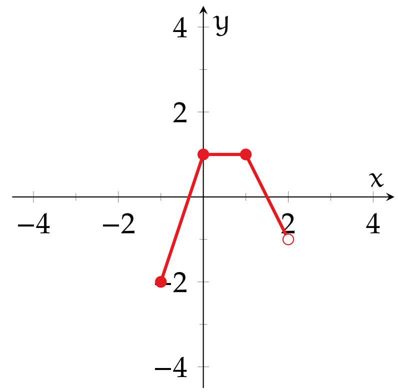
3. Shift up by 2



Graph of $2f(x+1) + 2$.

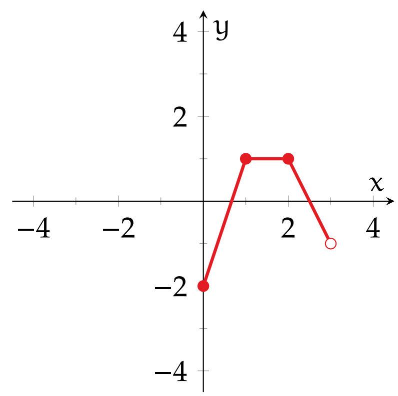
Similarly, we can find the graph $y = -f(2(x-1))$

1. Shrink horizontally by 2



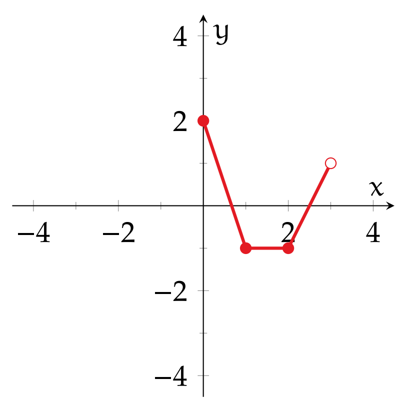
Graph of $f(2x)$.

2. Shift right by 1.



Graph of $f(2(x-1))$.

3. Reflect about the x-axis.



Graph of $-f(2(x-1))$.

We can conclude this section with the following table summarizing the discussed transformations.

Transformation	Appearance in Function	Transformation of point
Shift up	$f(x) \mapsto f(x) + k$	$(x, y) \mapsto (x, y + k)$
Shift down	$f(x) \mapsto f(x) - k$	$(x, y) \mapsto (x, y - k)$
Shift left	$f(x) \mapsto f(x + k)$	$(x, y) \mapsto (x - k, y)$
Shift right	$f(x) \mapsto f(x - k)$	$(x, y) \mapsto (x + k, y)$
Reflection about x-axis	$f(x) \mapsto -f(x)$	$(x, y) \mapsto (x, -y)$
Reflection about y-axis	$f(x) \mapsto f(-x)$	$(x, y) \mapsto (-x, y)$
Reflection about origin	$f(x) \mapsto -f(-x)$	$(x, y) \mapsto (-x, -y)$
Vertical scaling	$f(x) \mapsto kf(x)$	$(x, y) \mapsto (x, ky)$
Horizontal scaling	$f(x) \mapsto f(kx)$	$(x, y) \mapsto (\frac{x}{k}, y)$

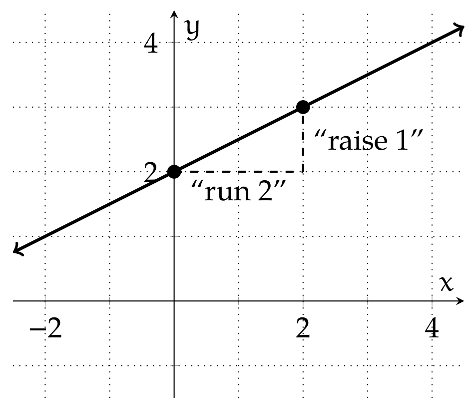
2.1 Linear Functions

Definition 2.1.1. (1) A function is *linear* if its graph is a line in the plane.

(2) The *slope* of a linear function is how much y differs per change of x.

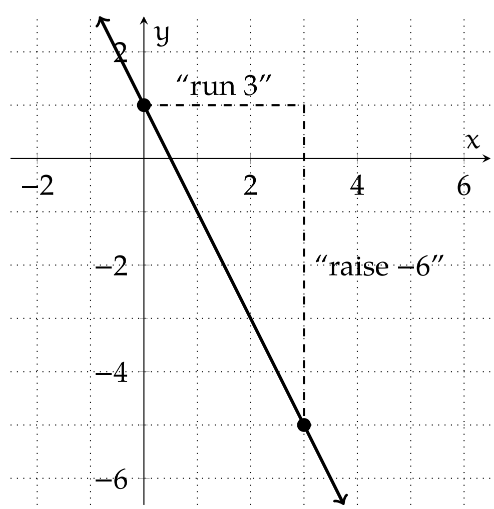
Example 2.1.2.

- The slope of a linear function provided the graph can be seen as "raise over run." Consider the function $f(x) = \frac{x}{2} + 2$.



We can see that the slope of the line is $\frac{1}{2}$. Every step on the x-axis, the line raises $\frac{1}{2}$ on the y-axis.

- The slope of the line $y = 1 - 2x$ is $\dfrac{-6}{3} = -2$:



- The slope of the line connecting (x_1, y_1) and (x_2, y_2) is $\dfrac{y_1 - y_2}{x_1 - x_2}$.

- Not every line has a slope. Consider the line connecting $(1, 3)$ and $(1, 2)$.

Exercise. Graph the following lines then find the slope.

a) The line through the points $(-2, 4)$ and $(3, -1)$

b) The horizontal line $y = 2$.

Fact 2.1.3.

(1) The line with slope m and y-intercept $(0, b)$ is given by the equation $y = mx + b$. This is called the *slope-intercept form*.

(2) The line with slope m through the point (a, b) is given by the equation $y - a = m(x - b)$. This is called the *point-slope form*.

Example 2.1.4. Consider the line through the point $(2, 1)$ with slope 5.

- The point-slope form is easy to find: $y - 1 = 5(x - 2)$.
- To find the slope-intercept form we first need to find the y-intercept.

$$y - 1 = 5(0 - 2) \implies y - 1 = 5(-2) \implies y - 1 = -10 \implies y = -9$$

Thus, the y-intercept is $(0, -9)$ and the slope-intercept form is $y = 5x - 9$.

Remark 2.1.5. Thus, given a point (a, b) and the slope m it is easy to set up an equation for a line. However, we need always given the slope. To write an equation for the line through the points (a_1, b_1) and (a_2, b_2) we first need to find the slope.

Example 2.1.6. Consider the line through the points $(-1, -3)$ and $(2, -6)$.

- The slope is found by $\dfrac{-3 - (-6)}{-1 - 2} = \dfrac{3}{-3} = -1$.
- Thus, point-slope forms are given by $y + 3 = -(x + 1)$ and $y - 2 = -(x + 6)$ depending on which point is your favorite.

Exercise. Give an equation for the following lines. (Any form is fine)

a) The line with slope $m = 2$ through the point $(-1, 2)$.

b) The line through the points $(3, 1)$ and $(1, 3)$.

Definition 2.1.7. Suppose f and g are linear functions with slopes m and n. Then f and g are

(1) *parallel* if $m = n$.

(2) *perpendicular* if $mn = -1$.

Example 2.1.8.

- $y = 2x + 1$ and $y - 4 = 2(x - 7)$ are parallel because they have the same slope.
- $y - 1 = 3x$ and $y = -\frac{1}{3}x + 5$ are perpendicular because $3(-\frac{1}{3}) = -1$.

- $y = 4x + 1$ and $y = -4x + 1$ are not parallel because they have a different slope, and not perpendicular because $4(-4) = -16 \neq -1$.

Remark 2.1.9. Two lines a perpendicular if their slopes are each others reciprocal with the opposite sign. For example, if f has slope $m = 2$ and g has slope $n = -\dfrac{1}{2}$ then $mn = 2(-\frac{1}{2}) = -1$, so they are perpendicular.

Example 2.1.10. Consider the line $f(x) = 3x + 1$.

- Let's find a linear function g that is parallel to f and goes through the point $(1, -1)$. By the definition we need the slope of g to be 3. Now we can apply the point-slope form to obtain an equation:

$$y + 1 = 3(x - 1) \quad \Longrightarrow \quad y = 3(x - 1) - 1 = 3x - 3 - 1 = 3x - 4$$

  so g can be defined by $g(x) = 3x - 4$.

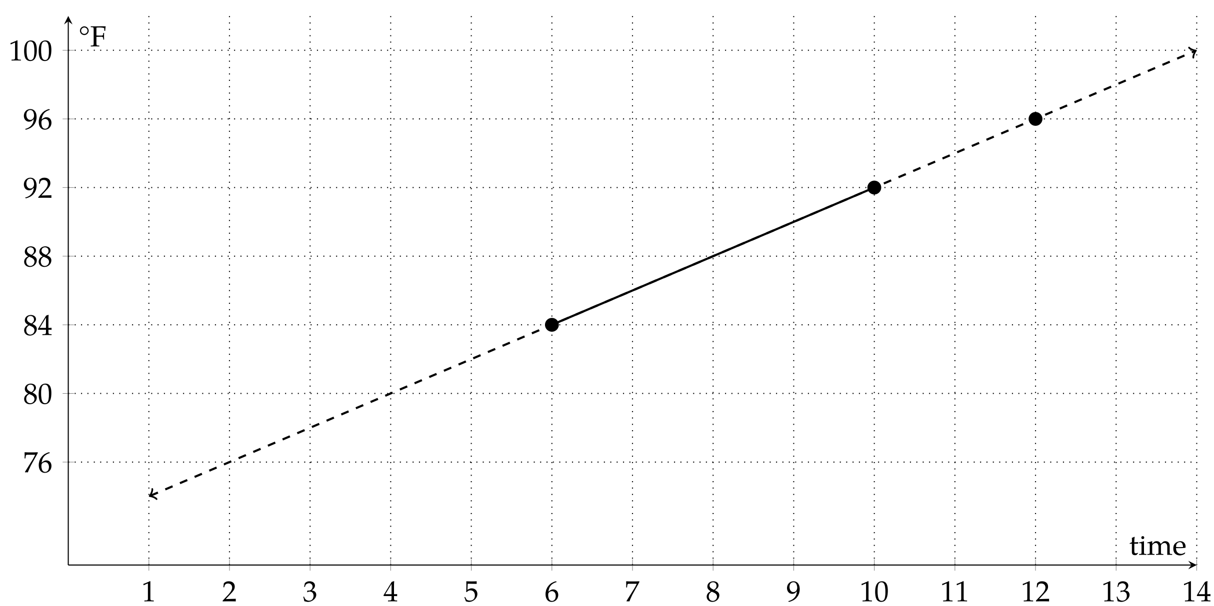
- To find a line that is perpendicular to f we need its slope to be the negative reciprocal of 3, so $-\frac{1}{3}$. If the line should have y-intercept $(0, 2)$ we can define it using slope-intercept form: $y = -\frac{1}{3}x + 2$.

Exercise. Find equations for the following lines.

a) The line that is parallel to the line $y = 2x - 1$, and passes through the point $(1, 7)$.

b) The line that is perpendicular to the line $y - 1 = \dfrac{x-2}{4}$, and passes through the point $(2, -3)$.

Application 2.1.11. Suppose that two separate temperature readings were taken at the the top of A-mountain: at 6 AM the temperature was 84°F and at 10 AM it was 92°F. Assuming the temperature changes linearly, we can graph the following picture.



- The slope of this line is $\frac{92-84}{10-6} = \frac{8}{4} = 2$.
- This means every hour the temperature increases by 2°F.
- We can set up the point-slope form and rewrite:

$$T(h) - 92 = 2(h - 10) \quad \Longrightarrow \quad T(h) = 2h - 20 + 92 \quad \Longrightarrow \quad T(h) = 2h + 72,$$

  this defines a function T which gives the temperature for hour h.

- Now we can predict the temperature for example at noon:

$$T(12) = 2(12) + 72 = 24 + 72 = 96°\text{F}$$

Application 2.1.12. Suppose at a local robotic factory it costs \$1000 to design a robot, and then an additional \$80 for each robot produced.

- Then the cost C, in dollars, to produce n robots is given by:

$$C(n) = 80n + 1000$$

- For example, to produce a line with 10 robots it costs

$$C(10) = 80(10) + 1000 = 1800\$.$$

- Say the company had a startup budget of \$15,000. We want to figure out how many robots we can produce for that, so we need to solve

$$\begin{aligned} C(n) &= 15000 \\ 80n + 1000 &= 15000 \\ 80n &= 14000 \\ n &= 175, \end{aligned}$$

  so for \$15,000 the company can produce at most 175 robots.

- Assuming each robot sells for \$100, we can calculate how many robots the company needs to sell to make a profit. Namely, a formula for the income is $P(n) = 100n$. Then we need to solve

$$\begin{aligned} C(n) &< P(n) \\ 80n + 1000 &< 100n \\ 1000 &< 20n \\ 50 &< n, \end{aligned}$$

  so they need to sell more than 50 robots to make a profit.

2.2 Absolute Value Function

Definition 2.2.1. For $x \in (-\infty, \infty)$, define the <u>*absolute value*</u> by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

Example 2.2.2.

- $|2| = 2$
  since $2 \geqslant 0$
- $|-12| = -(-12) = 12$
  since $-12 < 0$
- $|0| = 0$
  since $0 \geqslant 0$

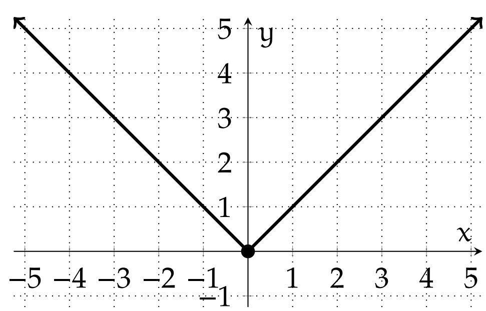
Exercise. Evaluate the following.

a) $|-11|$ b) $|42|$ c) $|-4 \cdot 5|$ d) $|-4| \cdot |5|$ e) $|(-3)^2|$ f) $|-3|^2$

Remark 2.2.3.

(1) One can see the $|x|$ as the distance between x and 0. For example, $|5| = 5$ since there are five units between 5 and 0. Similarly, $|-5|$ is the distance is between -5 and 0, which is again 5.

(2) The graph of $y = |x|$ is:



(3) The domain of the absolute value function is $(-\infty, \infty)$.

(4) The range of the absolute value function is $[0, \infty)$. It is **always** bigger or equal to 0.

Fact 2.2.4. Let $a, b \in (-\infty, \infty)$ and n an integer. Then

$$|ab| = |a| \cdot |b|$$
$$|a^n| = |a|^n \qquad \text{(need } a^n \text{ defined)}$$
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \qquad \text{(need } b \neq 0\text{).}$$

Example 2.2.5.

- $|-2x| = |-2| \cdot |x| = 2|x|$

- $|(-3)^3| = |-3|^3 = 3^3 = 27$
- $\left|\dfrac{x^2-1}{-10}\right| = \dfrac{|x^2-1|}{|-10|} = \dfrac{|x^2-1|}{10}$

Example 2.2.6. Consider $f(x) = \begin{cases} x-5 & \text{if } x < 0 \\ 2x & \text{if } 0 \leqslant x \leqslant 10 \\ \frac{x}{2} & \text{if } x > 10 \end{cases}$. Then

$$f(x) = 8 \quad \Longrightarrow \quad \begin{cases} x-5 = 8 & \text{if } x < 0 \\ 2x = 8 & \text{if } 0 \leqslant x \leqslant 10 \\ \frac{x}{2} = 8 & \text{if } x > 10 \end{cases}$$

Thus, there are three cases to solve:

$(x < 0)$:
$x - 5 = 8 \implies x = 13$
Extraneous.

$(0 \leqslant x \leqslant 10)$:
$2x = 8 \implies x = 4$

$(x > 10)$:
$\dfrac{x}{2} = 8 \implies x = 16$

Thus, the solutions are $x = 4$ and $x = 16$. ($x = 13$ is an extraneous solution)

Example 2.2.7. Solving equations involving absolute values is similar.

- $|x| = 42 \quad \Longrightarrow \quad \begin{cases} -x = 42 & \text{if } x < 0 \\ x = 42 & \text{if } x \geqslant 0 \end{cases}$

  $(x < 0)$: $-x = 42 \implies x = -42$ $\qquad$ $(x \geqslant 0)$: $x = 42$

  Solutions: $x = -42$ and $x = 42$

- $|x-3| = 5 \quad \Longrightarrow \quad \begin{cases} -(x-3) = 5 & \text{if } x-3 < 0 \\ x-3 = 5 & \text{if } x-3 \geqslant 0 \end{cases}$

  $(x-3 < 0)$: $-(x-3) = 5 \implies x - 3 = -5 \implies x = -2$ $\qquad$ $(x-3 \geqslant 0)$: $x - 3 = 5 \implies x = 8$

  Solutions: $x = -2$ and $x = 8$

- $|3x-1| - 6 = 0 \quad \Longrightarrow \quad |3x-1| = 6 \quad \Longrightarrow \quad \begin{cases} -(3x-1) = 6 & \text{if } 3x-1 < 0 \\ 3x-1 = 6 & \text{if } 3x-1 \geqslant 0 \end{cases}$

$(3x - 1 < 0)$:

$$-(3x-1) = 6 \implies 3x - 1 = -6 \implies 3x = -5 \implies x = \frac{-5}{3}$$

$(3x - 1 \geqslant 0)$:

$$3x - 1 = 6 \implies 3x = 7 \implies x = \frac{7}{3}$$

Solutions: $x = -\frac{5}{3}$ and $x = \frac{7}{3}$

- $10 - |1 - x| = 21 \implies -|1-x| = 11 \implies |1-x| = -11 \implies \begin{cases} -(1-x) = -11 & \text{if } 1 - x < 0 \\ 1 - x = -11 & \text{if } 1 - x \geqslant 0 \end{cases}$

$(1 - x < 0)$:

$$-(1-x) = -11 \implies 1 - x = 11 \implies -x = 10 \implies x = -10$$

Extraneous.

$(1 - x \geqslant 0)$:

$$1 - x = -11 \implies -x = -12 \implies x = 12$$

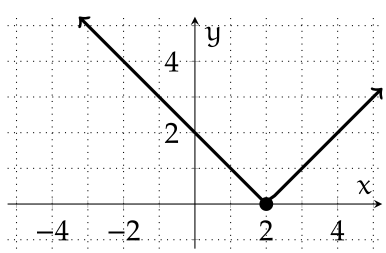
Extraneous.

Solutions: none.

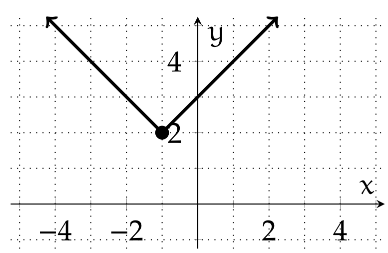
Exercise. Solve the following equations

a) $|x+3| = 5$ b) $|x| = -2$ c) $|2x+1| = 9$ d) $3 - |x| = 1$ e) $|7x-1|-2 = 0$

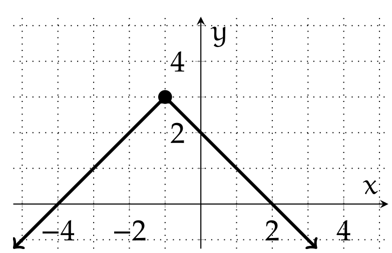
Example 2.2.8. We can use transformations to graph more complicated functions using absolute values.



Graph of $f(x) = |x-2|$



Graph of $g(x) = |x+1| + 2$



Graph of $h(x) = -|x+1| + 3$

From the functions we can now now easily get the domain/range, intercepts, zeros, extrema, etc.

Function	$f(x)$	$g(x)$	$h(x)$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[0, \infty)$	$[2, \infty)$	$(-\infty, 3]$
x-intercept(s)	$(2, 0)$	DNE	$(-4, 0)$ and $(2, 0)$
y-intercept	$(0, 2)$	$(0, 3)$	$(0, 2)$
Interval(s) of increase	$[2, \infty]$	$[-1, \infty)$	$(-\infty, -1]$
Interval(s) of decrease	$(-\infty, 2]$	$(-\infty, -1]$	$[-1, \infty)$
Maximum	DNE	DNE	3
Minimum	0	2	DNE

Exercise. Graph the following functions.

a) $f(x) = |x + 2|$ b) $g(x) = -|x|$ c) $h(x) = |x| + 1$

Then find their domain, range, intercepts, (local) maxima/minima, intervals of increase/decrease, zeros, etc.

2.3 Quadratic Functions

Definition 2.3.1. A function $f(x)$ is *quadratic* if it can be written in the form

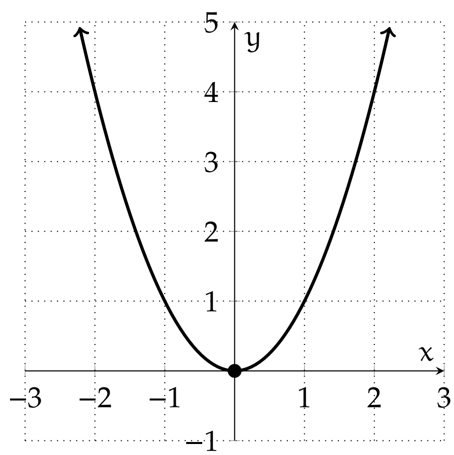
$$f(x) = ax^2 + bx + c,$$

where $a, b, c \in (-\infty, \infty)$ and $a \neq 0$. This is called the *general form*.

Example 2.3.2.

- $f(x) = x^2$ is a quadratic function
- $f(x) = 4x^2 - 1$ is a quadratic function
- $f(x) = -x^2 + x + 100$ is a quadratic function
- $f(x) = 3x - 4$ is not a quadratic function (it is a linear function)
- $f(x) = x^3$ is not a quadratic function
- $f(x) = 2(x-1)^2 + 3$ is a quadratic function (rewrite it to $f(x) = 2x^2 - 4x + 5$)

Remark 2.3.3. Recall, the graph of $f(x) = x^2$ is a parabola.



Every quadratic function can be obtained via transformations on $f(x) = x^2$. Consequently, the graph of every quadratic function is a parabola. Hence, every quadratic function has an increasing interval and a decreasing interval. Consequently, every quadratic function has a unique local minimum/maximum.

Definition 2.3.4. Let $f(x)$ be a quadratic function.

(1) The *vertex* of f is the local minimum/maximum of f.

(2) The *axis of symmetry* of f is the vertical line through the vertex of f.

Example 2.3.5.

- The vertex of $f(x) = x^2$ is $(0, 0)$, and the axis of symmetry is $x = 0$.
- The vertex of $g(x) = (x + 2)^2$ is $(-2, 0)$ (left shift by 2), and the axis of symmetry is $x = -2$.
- The vertex of $h(x) = -4(x + 2)^2$ is $(-2, 0)$ (vertical scale -4), and the axis of symmetry is $x = -2$.
- The vertex of $k(x) = -4(x+2)^2+3$ is $(-2, 3)$ (up shift by 3), and the axis of symmetry is $x = -2$.
- The vertex of $\ell(x) = a(x + h)^2 + k$ is $(-h, k)$, and the axis of symmetry is $x = -h$.

Definition 2.3.6. The *standard form* (or *vertex form*) of a quadratic function f is

$$f(x) = a(x + h)^2 + k.$$

Fact 2.3.7. Let $f(x) = a(x + h)^2 + k$ be a quadratic function in standard form.

(1) $(-h, k)$ is the vertex of f.

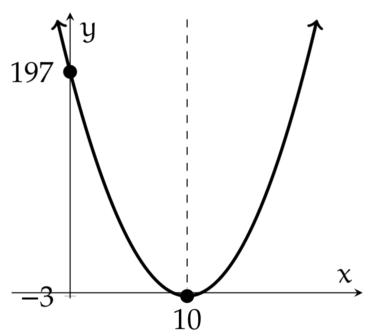
(2) $x = -h$ is the axis of symmetry of f.

(3) If $a > 0$ then the graph of f is an upward opening parabola $\bigcup$-shape.

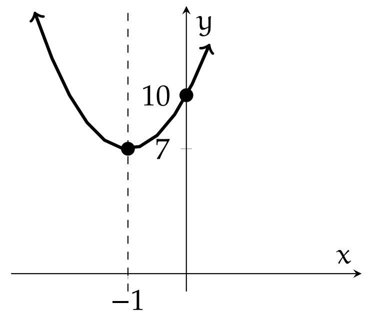
(4) If $a < 0$ then the graph of f is a downward opening parabola $\bigcap$-shape

Example 2.3.8.

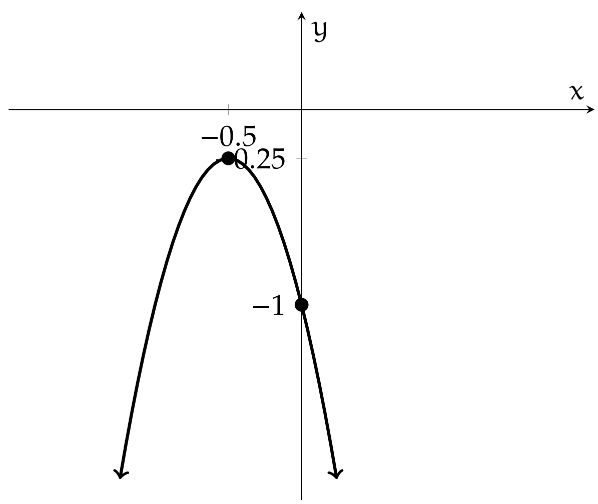
- The vertex of $f(x) = 2(x - 10)^2 - 3$ is $(10, -3)$, the axis of symmetry is $x = 10$and $y = f(x)$ is an upward opening parabola:



- The vertex of $g(x) = 3(x + 1)^2 + 7$ is $(-1, 7)$, the axis of symmetry is $x = -1$, and $y = g(x)$ is an upward opening parabola:



- The vertex of $h(x) = -3(x + \frac{1}{2})^2 - \frac{1}{4}$ is $(-\frac{1}{2}, -\frac{1}{4})$, the axis of symmetry is $x = -\frac{1}{2}$ and $y = h(x)$ is a downward opening parabola:



Exercise. Find the vertex of $f(x) = 3(x + 1)^2 + 10$

Exercise. Graph $h(x) = -(x - 1)^2 + 2$.

Example 2.3.9 (Completing the square)**.**

- Let $f(x) = x^2 - 2x + 3$. To find the vertex of f, we need to convert f into standard form. Observe,

$$\begin{aligned} f(x) &= x^2 - 2x + 3 \\ &= x^2 - 2x + 1 - 1 + 3 \\ &= (x^2 - 2x + 1) - 1 + 3 \\ &= (x - 1)(x - 1) - 1 + 3 \\ &= (x - 2)^2 + 2 \end{aligned}$$

  This method is called *completing the square*; we turned $x^2 - 2x$ into a square by adding 1. We can now see that the vertex of $f(x)$ is $(2, 2)$.

- Let $g(x) = x^2 + 6x + 1$. Observe,

$$\begin{aligned} g(x) &= x^2 + 6x + 1 \\ &= (x^2 + 6x + 9) - 9 + 1 \\ &= (x + 3)^2 - 8, \end{aligned}$$

  so the vertex is $(-3, 8)$.

- Let $h(x) = 2x^2 + 8x + 10$. Observe,

$$\begin{aligned} h(x) &= 2x^2 + 8x + 10 \\ &= 2(x^2 + 4x + 5) \\ &= 2(x^2 + 4x + 4 - 4 + 5) \\ &= 2((x^2 + 4x + 4) + 1) \\ &= 2((x + 2)^2 + 1) \\ &= 2(x + 2)^2 + 2, \end{aligned}$$

  so the vertex is $(-2, 2)$.

- Let $p(x) = -3x^2 - 2x + 8$. Observe,

$$\begin{aligned} p(x) &= -3x^2 - 2x + 8 \\ &= -3\left(x^2 - \frac{2x}{-3} + \frac{8}{-3}\right) \\ &= -3\left(x^2 + \frac{2}{3}x - \frac{8}{3}\right) \\ &= -3\left(\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{9} - \frac{8}{3}\right) \\ &= -3\left(\left(x + \frac{1}{9}\right)^2 - \frac{1}{9} - \frac{8}{3}\right) \\ &= -3\left(\left(x + \frac{1}{9}\right)^2 - \frac{1}{9} - \frac{24}{9}\right) \\ &= -3\left(\left(x + \frac{1}{9}\right)^2 - \frac{25}{9}\right) \\ &= -3\left(x + \frac{1}{9}\right)^2 + (-3) \cdot \frac{25}{9} \\ &= -3\left(x + \frac{1}{9}\right)^2 + \frac{25}{3}, \end{aligned}$$

  so the vertex is $(-1/9, 25/3)$.

Exercise. Let $g(x) = x^2 - 8x + 11$. Find the vertex form of $g(x)$ by completing the square.

Example 2.3.10.

- Let $f(x) = 2(x+1)^2 - 8$. To find the x-intercepts, set $f(x) = 0$.

$$\begin{aligned} 2(x+1)^2 - 8 = 0 \quad &\implies \quad 2(x+1)^2 = 8 \\ &\implies \quad (x+1)^2 = 4 \\ &\implies \quad x + 1 = \sqrt{4} \qquad \text{or} \qquad x + 1 = -\sqrt{4} \\ &\implies \quad x + 1 = 2 \qquad \text{or} \qquad x + 1 = -2 \\ &\implies \quad x = 1 \qquad \text{or} \qquad x = -3 \end{aligned}$$

  Thus, the x-intercepts are $(-3, 0)$ and $(1, 0)$.

- Let $g(x) = x^2 + 3x + \frac{1}{4}$. Observe, it sis hard to find the x-intercepts in the usual way:

$$x^2 + 3x + \frac{1}{4} = 0 \quad \implies \quad ?$$

  We can use the completing square method to rewrite the LHS:

$$\begin{aligned} x^2 + 3x + \frac{1}{4} &= x^2 + 3x + \frac{9}{4} - \frac{9}{4} + \frac{1}{4} \\ &= \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + \frac{1}{4} \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{1}{4} \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{8}{4} \\ &= \left(x + \frac{3}{2}\right)^2 - 2, \end{aligned}$$

  Now we can find the x-intercepts:

$$\begin{aligned} x^2 + 3x + 1 = 0 \quad &\implies \quad \left(x + \frac{3}{2}\right)^2 - 2 = 0 \\ &\implies \quad \left(x + \frac{3}{2}\right)^2 = 2 \\ &\implies \quad x + \frac{3}{2} = \sqrt{2} \qquad \text{or} \qquad x + \frac{3}{2} = -\sqrt{2} \\ &\implies \quad x = \sqrt{2} - \frac{3}{2} \qquad \text{or} \qquad x = -\sqrt{2} - \frac{3}{2}, \end{aligned}$$

  so the x-intercepts are $(\sqrt{2} - \frac{3}{2}, 0)$ and $(-\sqrt{2} - \frac{3}{2}, 0)$. Recall, this means that $x = \sqrt{2} - \frac{3}{2}$ and $x = -\sqrt{2} - \frac{3}{2}$ are the zeros of $g(x)$.

Fact 2.3.11 (Quadratic formula)**.** Let $f(x) = ax^2 + bx + c$.

(1) The zeros of $f(x)$ are
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$
provided $b^2 - 4ac \geqslant 0$.

(2) If $b^2 - 4ac < 0$ then $f(x)$ has no zeros.

Example 2.3.12.

- Let $g(x) = x^2 + 3x + \frac{1}{4}$. Then the zeros of $g(x)$ are
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(1)(\frac{1}{4})}}{2(1)} \\ &= \frac{-3 \pm \sqrt{9 - 1}}{2} \\ &= \frac{-3 \pm \sqrt{8}}{2} \\ &= \frac{-3 \pm 2\sqrt{2}}{2} \\ &= \frac{3}{2} \pm \sqrt{2}, \end{aligned}$$
i.e., we found the same zeros as in the previous example.

- Let $h(x) = x^2 + 1$. Then the zeros of $h(x)$ are
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} \\ &= \frac{\pm\sqrt{0 - 4}}{2} \\ &= \frac{\pm\sqrt{-4}}{2} \\ &= \text{oops}. \end{aligned}$$
The first part of the fact does not apply since $b^2 - 4ac = 0^2 - 4(1)(1) = -4 < 0$. By the second part, $h(x)$ has no zeros.

3.1 Graphs of Polynomials

The aim of this section is to allows us to sketch the graph of any polynomial.

Definition 3.1.1. A *polynomial function* is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $n \geqslant 0$ and all $a_i \in (-\infty, \infty)$. The domain of a polynomial function is $(-\infty, \infty)$.

Fact 3.1.2. Let $f(x)$ be a function. If $\text{dom}(f) \neq (-\infty, \infty)$ then $f(x)$ is not a polynomial.

Example 3.1.3.

Polynomials	**Not Polynomials**
$f(x) = x^3 - 3x^2 + 2x - 5$	
$f(x) = 4x + 7x^2 - 2 - 10x^3 + x^7$	$f(x) = \dfrac{1}{x}$
$f(x) = 42$	$f(x) = \sqrt{x}$
$f(x) = 3x - 10$	$f(x) = \|x\|$
$f(x) = 3x^2 - 2$	
$f(x) = \dfrac{4x + x^3}{4}$	
$f(x) = (x+1)(x-3)^2$	

Definition 3.1.4. Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial function with $a_n \neq 0$.

- The *leading term* of f is the term $a_n x^n$.
- The *degree* of f is the natural number n.
- The *leading coefficient* of f is the real number a_n.
- The *constant term* of f is the real number a_0.

If $f(x) = 0$ then we say f has *no degree*.

Example 3.1.5.

- $f(x) = 4x^5 - 3x^2 + 2x - 5$:

  leading term: $4x^5$, degree: 5, leading coefficient: 4, constant term: -5.

- $g(x) = 12x + x^3$:

  leading term x^3, degree 3, leading coefficient: 1, constant term: 0.

- $h(x) = \dfrac{4-x}{5}$:

  leading term $-\frac{1}{5}x$, degree: 1, leading coefficient: $-\frac{1}{5}$, constant term: $\frac{4}{5}$.

- $p(x) = (2x-1)^3(x-2)(3x+2)$:
  - leading term $(2x)^3(x)(x)(3x) = 24x^5$
  - degree: 5
  - leading coefficient 24
  - constant term: $(-1)^3(-2)(2) = 4$
- $q(x) = (3-2x)^2(1+x)^{10}(x^2-4)$
  - leading term $(-2x)^2(x)^{10}(x)(x^2) = (4x^2)(x^{10})(x^2) = 4x^14$
  - degree: 14
  - leading coefficient: 4
  - constant term: $(3)^2(1)^{10}(-4) = (9)(1)(-4) = -36$

Exercise. Find (i) the leading term, (ii) the degree, (iii) the leading coefficient, and (iv) the constant term of the given polynomial.

a) $f(x) = x^4 - 10x^2 + 1$

b) $g(x) = 42 - x^3$

c) $h(x) = \dfrac{x^2-4}{2}$

d) $p(x) = (x-1)^9(x-2)(x-3)(x-4)$

Fact 3.1.6. Let $f(x)$ be a polynomial and $a \in (-\infty, \infty)$. The following are equivalent.

(1) a is the constant term of f.

(2) $f(0) = a$.

(3) $(0, a)$ is the y-intercept of the graph of f.

Remark 3.1.7.

(1) Since the constant term gives us the y-intercept, it is useful for sketching the graph of a polynomial.

(2) Recall that to find the y-intercept of a function $f(x)$, we just have to evaluate $f(0)$. Hence, we can also find the constant term like this.

Example 3.1.8.

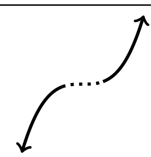
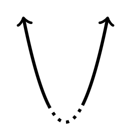
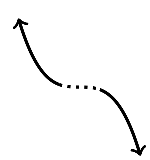
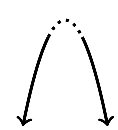
- Suppose $f(x) = 2(x+1)^2(2-x)^3$. Then $f(0) = 2(0+1)^2(2-0)^3 = 2(1)^2(2)^3 = 2(1)(8) = 16$, so the constant term is 16 and the y-intercept is $(0, 16)$.
- Suppose $g(x)$ is a polynomial with y-intercept $(0, -11)$. Then the constant term of f is -11 and $g(0) = -11$.
- Suppose $h(x)$ is a polynomial such that $h(0) = \sqrt{2}$. Then the constant term of g is $\sqrt{2}$ and the y-intercept is $(0, \sqrt{2})$.

Definition 3.1.9 (End behavior)**.** The *end behavior* of a function $f(x)$ is a description of what is happening to the function values $f(x)$ as the x-values go towards $-\infty$ and ∞.

Example 3.1.10.

- The end behavior of $f(x) = x^2$ can be described as follows:
  - if $x \to \infty$ then $f(x) \to \infty$.
  - if $x \to -\infty$ then $f(x) \to \infty$.
- The end behavior of $g(x) = 2x - 1$ can be described by:
  - if $x \to \infty$ then $g(x) \to \infty$.
  - if $x \to -\infty$ then $g(x) \to -\infty$.
- The end behavior of $h(x) = 42$ can be described by:
  - if $x \to \infty$ then $h(x) \to 42$.
  - if $x \to -\infty$ then $h(x) \to -42$.
- The end behavior of $j(x) = \frac{1}{x}$ can be described by:
  - if $x \to \infty$ then $j(x) \to 0$.
  - if $x \to -\infty$ then $j(x) \to 0$.

Fact 3.1.11. The end behavior of a polynomial $f(x)$ is determined by it's leading coefficient and degree.

	Odd degree	**Even degree**
Positive leading coefficient	 $x \to \infty \implies f(x) \to \infty$ $x \to -\infty \implies f(x) \to -\infty$	 $x \to \infty \implies f(x) \to \infty$ $x \to -\infty \implies f(x) \to \infty$
Negative leading coefficient	 $x \to \infty \implies f(x) \to -\infty$ $x \to -\infty \implies f(x) \to \infty$	 $x \to \infty \implies f(x) \to -\infty$ $x \to -\infty \implies f(x) \to -\infty$

Remark 3.1.12.

(1) It is enough to know the leading term of a polynomial to determine its end behavior.

(2) Knowing the end behavior is essential to graph the sketch of a function.

Example 3.1.13.

- The end behavior of $f(x) = 4x^{10} - x^9 + 10$ can be described by:
  - if $x \to \infty$ then $f(x) \to \infty$.
  - if $x \to -\infty$ then $f(x) \to \infty$.
- The end behavior of $g(x) = -2x^5 - 1$ can be described by:
  - if $x \to \infty$ then $g(x) \to -\infty$.
  - if $x \to -\infty$ then $g(x) \to \infty$.
- The end behavior of $h(x) = (x-5)^2(2-x)^3(x+1)^{19}$ can be described by:
  - if $x \to \infty$ then $h(x) \to -\infty$.
  - if $x \to -\infty$ then $h(x) \to -\infty$.

  Leading term: $(x)^2(-x)^3(x)^{19} = (x^2)(-x^3)(x^{19}) = -x^{24}$, so the degree is even and the leading coefficient is negative.

Exercise. Suppose $f(x) = (x-1)^3(x+2)$. Find/evaluate

a) leading term
b) degree
c) leading coefficient
d) constant term
e) y-intercept
f) end behavior

Remark 3.1.14.

(1) The end behavior tells us what the graph of a function does at the "ends." However, except for the y-intercept we do not really know what the graph does in between. This is where we will use x-intercepts of the functions. For this we need to know how a polynomial behaves at it's x-intercepts.

(2) Recall that $(c, 0)$ is an x-intercept of $f(x)$ if and only if $f(c) = 0$ if and only if $x = c$ is a zero of $f(x)$.

Fact 3.1.15. Let $a, b \in (-\infty, \infty)$. If $ab = 0$ then $a = 0$ or $b = 0$.

Remark 3.1.16. You can only split factors like this if they equal 0. It is not the case that if $ab = c$ then $a = c$ or $b = c$. For example, suppose $ab = 6$ then $a = 2 \neq 6$ and $b = 3 \neq 6$.

Example 3.1.17. Let $f(x) = (x+2)(x-3)$. Then zeros of f are found by solving $f(x) = 0$:

$$\begin{aligned} (x+2)(x-3) = 0 \quad &\Longrightarrow \quad (x+2) = 0 \text{ or } (x-3) = 0 \\ &\Longrightarrow \quad x = -2 \text{ or } x = 3 \end{aligned}$$

However, we cannot solve $f(x) = 6$ in this way:

$$(x+2)(x-3) = 6 \;\not\Longrightarrow\; x+2 = 6 \text{ or } x-3 = 6$$

Definition 3.1.18. Suppose $f(x)$ is a polynomial function and $c \in (-\infty, \infty)$ is a zero of f. The *multiplicity* of c is the number of times $(x - c)$ is a factor of $f(x)$.

Example 3.1.19.

- Let $f(x) = (x + 2)(x - 1)^3$. The zeros and their multiplicity are found using the factors $(x + 2)$ and $(x - 1)$. Namely, $x = -2$ is a zero of multiplicity 1 and $x = 1$ is a zero of multiplicity 3.
- Let $g(x) = (2x - 4)^9(x + 3)^3$. Then $x = 2$ is a zero of multiplicity 9 and $x = -3$ is a zero of multiplicity 3.
- Let $h(x) = x^2 - 2x - 8$. Factoring gives $h(x) = (x - 4)(x + 2)$, so the zeros are $x = 4$ and $x = -2$ both with multiplicity 1.
- Let $p(x) = x^3 + 3x^2 - 9x - 27$. Not always easy to find the zeros.

Exercise. Tind zeros and their multiplicity of the following polynomials

a) $f(x) = (x + 1)^2(x - 4)^3$ b) $g(x) = (x - 3)(2x - 9)^4$ c) $p(x) = x^2 - 9$

Fact 3.1.20. Suppose $f(x)$ is a polynomial function and $x = c$ is a zero of multiplicity m.

- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.
- If m is large then the angle $f(x)$ approaches $(c, 0)$ at is small.

Example 3.1.21.

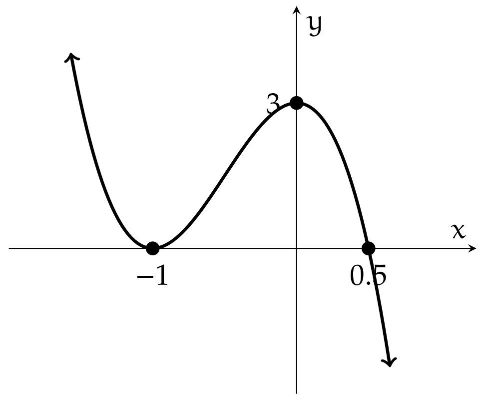
- Let $f(x) = -3(2x-1)(x+1)^2$. We will sketch $y = f(x)$. The leading term of $f(x)$ is $(-3)(2x)(x)2 = -6x^3$. Thus, f has odd degree and a negative leading coefficient. Hence, end behavior is described by:

  - If $x \to -\infty$ then $f(x) \to \infty$.
  - If $x \to \infty$ then $f(x) \to -\infty$.

  To find the x-intercepts, we will find the zeros: $f(x) = 0$

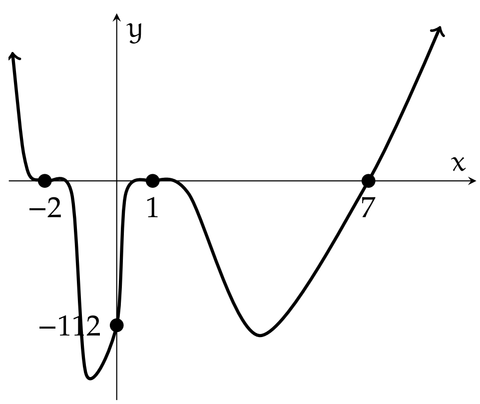
  - $2x - 1 = 0 \iff 2x = 1 \iff x = \frac{1}{2}$ with multiplicity 1
  - $x + 1 = 0 \iff x = -1$ with multiplicity 2

  Hence, x-intercepts are $(\frac{1}{2}, 0)$ and $(-1, 0)$.



The constant term is $f(0) = -3(2 \cdot 0 - 1)(0+1)^2 = -3(-1)(1)^2 = 3(1) = 3$, so the y-intercept is $(0, 3)$.

- Suppose $g(x) = (x-1)^4(x+2)^3(2x-14)$.
  - Leading term: $(x)^4(x)^3(2x) = 2x^8$.
  - Degree: 8 (even)
  - Leading coefficient: 2 (positive)
  - Constant term: $f(0) = (-1)^4 \cdot 2^3 \cdot -14 = 1 \cdot 8 \cdot (-14) = -112$.
  - End behavior: if $x \to -\infty$ then $f(x) \to \infty$, and if $x \to \infty$ then $f(x) \to \infty$.
  - Zeros: $x = 1$ with multiplicty 4, $x = -2$ with multiplicity 3, $x = 7$ with multiplicity 1
  - x-intercepts: $(1, 0), (-2, 0), (7, 0)$
  - y-intercept: $(-112, 0)$



Exercise. Sketch the graph of $f(x) = -(x+1)^5(x-2)^2(x-5)$.

3.2 Factor Theorem and Remainder Theorem

Remark 3.2.1. Recall, to sketch the graph of a polynomial it is important to be able to find the zeros. For this purpose we want to factor polynomials.

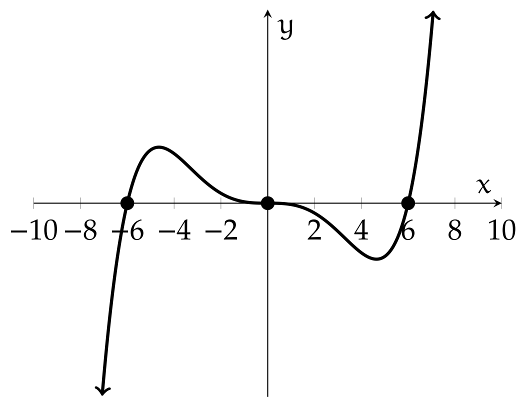
Example 3.2.2.

- Let $f(x) = x^5 - 36x^3$. The leading term of $f(x)$ is x^5, so the end behavior is
  - if $x \to -\infty$, then $f(x) \to -\infty$
  - if $x \to \infty$, then $f(x) \to \infty$

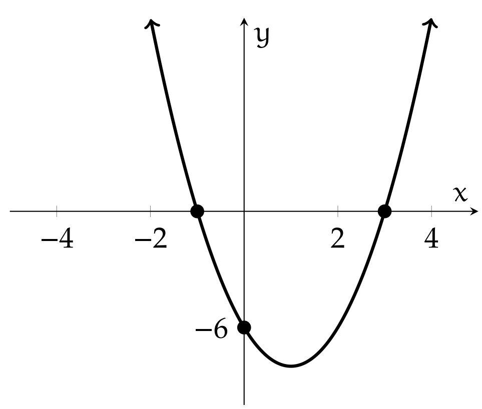
  The constant term is 0, so the y-intercept is $(0, 0)$. To find the x-intercepts we need to factor:

  $$x^5 - 36x^3 = x^3(x^2 - 36) = x^3(x - 6)(x + 6).$$

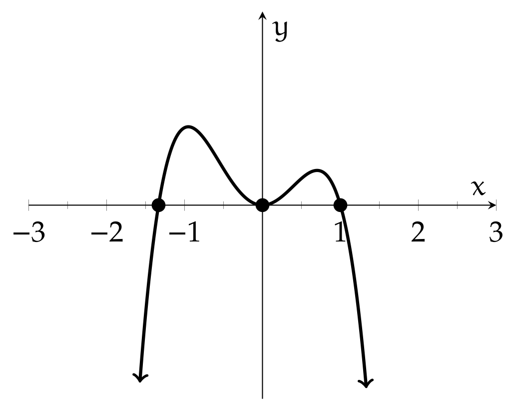
  Thus, the zeros are 0 (multiplicity 3), -6 (multiplicity 1), and 6 (multiplicity 1). We can now sketch the graph of f:



- Let $g(x) = 2x^2 - 4x - 6$.
  - The leading term is $2x^2$, so the end behavior is ↖ ↗.
  - The constant term is -6, so the y-intercept is $(0, -6)$.
  - Factoring: $2x^2 - 4x - 6 = 2(x^2 - 2x - 3) = 2(x - 3)(x + 1)$ gives the zeros are 3 and -1 both with multiplicity 1.



- Let $h(x) = -3x^4 - x^3 + 4x^2$
  - The leading term is $-3x^4$, so the end behavior is $\swarrow\ddot{}\searrow$.
  - The constant term is 0, so the y-intercept is $(0, 0)$.
  - Factoring: $$\begin{aligned}-3x^4 - x^3 + 4x^2 &= (-x^2)(3x^2 + x - 4)\\ &= -(x)^2(3x^2 + 4x - 3x - 4)\\ &= -(x)^2\big((x)(3x+4) - (3x+4)\big)\\ &= -(x)^2(x-1)(3x+4)\end{aligned}$$
    so the zeros are 0 (multiplicity 3), 1 (multiplicity 1), $-\frac{4}{3}$ (multiplicity 1).



Exercise. Factor the following polynomials.

a) $f(x) = x^2 - 2x - 15$ b) $g(x) = 3x^2 - 27$ c) $h(x) = 2x^2 + x - 3$ d) $p(x) = x^3 + 3^2 + 9x$

Fact 3.2.3. Suppose $f(x)$ is a (nonzero) polynomial and $c \in (-\infty, \infty)$. The following are equivalent.

(1) $(x - c)$ is a factor of $f(x)$.

(2) c is a zero of f.

(3) $f(c) = 0$.

(4) $x = c$ is a solution to the equation $f(x) = 0$.

(5) $(c, 0)$ is an x-intercept of the graph $y = f(x)$.

Example 3.2.4. Let $f(x)$ be a polynomial.

- If $(x - 2)$ is a factor of $f(x)$ then $f(2) = 0$.
- If $(-1, 0)$ is an x-intercept of f then $(x + 1)$ is a factor f.
- If $f(5) \neq 0$ then $(x - 5)$ is not a factor of $f(x)$.
- If $(3, 1)$ is on the graph of f then $(x - 3)$ is not a factor of $f(x)$.

Remark 3.2.5.

(1) Recall, we can divide numbers using *long division*:

$$
\begin{array}{r}
448 \\
7\,\overline{)\,3141} \\
\underline{28} \\
34 \\
\underline{28} \\
61 \\
\underline{56} \\
5
\end{array}
$$

Thus, $3141 = 7(448) + 5$.

(2) Similarly, we can divide polynomials:

$$
\begin{array}{rrrrr}
 & 5x^2 & +13x & +39 & \\
\cline{2-5}
x-3\,\big) & 5x^3 & -2x^2 & & +1 \\
 & -5x^3 & +15x^2 & & \\
\cline{2-3}
 & & 13x^2 & & \\
 & & -13x^2 & +39x & \\
\cline{3-4}
 & & & 39x & +1 \\
 & & & -39x & +117 \\
\cline{4-5}
 & & & & 118
\end{array}
$$

Thus, $5x^3 - 2x^2 + 1 = (x-3)(x^2 - 12x + 36) + 118$.

(3) Polynomial long division allows us to factor polynomials if we know one of the factors/zeros.

Example 3.2.6. Let $f(x) = 3x^3 + 3x^2 - 30x + 24$ be a polynomial. Observe that $f(-4) = 0$, so -4 is a zero, and hence $(x+4)$ is factor. Then

$$
\begin{array}{rrrrr}
 & 3x^2 & -9x & +6 & \\
\cline{2-5}
x+4\,\big) & 3x^3 & +3x^2 & -30x & +24 \\
 & -3x^3 & -12x^2 & & \\
\cline{2-3}
 & & -9x^2 & -30x & \\
 & & 9x^2 & +36x & \\
\cline{3-4}
 & & & 6x & +24 \\
 & & & -6x & -24 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

$$\begin{aligned} \text{Thus, } f(x) = 3x^3 + 3x^2 - 30x + 24 &= (x+4)(3x^2 - 9x + 6) + 0 \\ &= 3(x+4)(x^2 - 3x + 2) \\ &= 3(x+4)(x-2)(x-1) \end{aligned}$$

Exercise. Use polynomial long division to perform the indicated division.

a) $(4x^2 + 3x - 1) \div (x - 3)$

b) $(2x^3 - x + 1) \div (x^2 + x + 1)$

Exercise. Given the polynomial expression and one of its zeros, find the remaining zeros and factor completely.

a) $x^3 - 24x^2 + 192x - 512$, $x = 8$.

b) $x^4 - 6x^3 - 16x^2 + 150x - 225$, $x = 3$ (multiplicity 2).

4.1 Introduction to Rational Functions

Definition 4.1.1. A *rational function* is a function of the form

$$f(x) = \frac{e(x)}{d(x)}$$

where e and d are polynomials.

Example 4.1.2.

- $f(x) = \frac{1}{x}$ is a rational function.
- $g(x) = \frac{x^2 - 9}{x - 3}$ is a rational function.
- $h(x) = x^3 + 42$ is a rational function; every polynomial is a rational function.
- $A(x) = |x|$ is not a rational function.
- $r(x) = \frac{1}{\sqrt{x+2}}$ is not a rational function.
- $p(x) = \frac{(x-2)(x+4)}{(x-4)(x-2)(x+1)}$ is a rational function.

Remark 4.1.3.

(1) *Rational numbers* are numbers of the form $\frac{a}{b}$ where a, b are integers, so rational functions are obtained by applying this idea to polynomials.

(2) Our aim is to be able to sketch rational functions.

Exercise. Which of the following functions are rational functions?

a) $f(x) = 42$ b) $g(x) = |4x+1|$ c) $h(x) = \frac{x^4}{x^3+1}$ d) $j(x) = \begin{cases} 1 - x & \text{if } x \leqslant 0 \\ 2x + 4 & \text{if } x > 0 \end{cases}$

Definition 4.1.4. We say a rational function $f(x) = \frac{e(x)}{d(x)}$ is in *lowest terms* (*simplest terms*) if $e(x)$ and $d(x)$ have no common factors.

Example 4.1.5.

- $f(x) = \frac{(x+1)(x+2)}{(x+3)(x-4)}$ is in lowest terms.
- $g(x) = \frac{(x+1)(x+2)}{(x+1)(x-4)}$ is not in lowest terms.

- $h(x) = \dfrac{x^2 - 9}{x + 3}$ is not in lowest terms.

- $i(x) = \dfrac{2(x + 1)}{4(x - 1)}$ is not in lowest terms.

- $j(x) = \dfrac{x^3 - 2x^2 - 13x - 10}{x + 2}$ is not in lowest terms (use polynomial long division).

- $k(x) = \dfrac{x^3 - 3x^2 - 10x - 24}{2x^3 + 2x^2 - 50x - 50}$ – not always easy to decide.

Exercise. Reduce $f(x) = \dfrac{x^2 - 16}{2x + 8}$ to its lowest terms.

Remark 4.1.6. Since rational functions can have division in them, their domain is not always everything.

Example 4.1.7.

- Let $f(x) = \dfrac{2x - 1}{x + 1}$.

  Need $x + 1 \neq 0 \implies x \neq -1 \implies \text{dom}(f) = (-\infty, -1) \cup (-1, \infty)$.

- Let $g(x) = 2 - \dfrac{3}{x + 1}$.

  Need $x + 1 \neq 0 \implies x \neq -1 \implies \text{dom}(g) = (-\infty, -1) \cup (-1, \infty)$.

- Let $h(x) = \dfrac{2x^2 - 1}{x^2 - 1} - \dfrac{3x - 2}{x^2 - 1}$.

  $$\begin{aligned} \text{Need } x^2 - 1 \neq 0 \implies (x - 1)(x + 1) \neq 0 &\implies x \neq \pm 1 \\ &\implies \text{dom}(h) = (-\infty, -1) \cup (-1, 1) \cup (1, \infty). \end{aligned}$$

- Let $r(x) = \dfrac{\left(\dfrac{2x - 1}{x^2 - 4}\right)}{\left(\dfrac{x + 1}{x^2 - 4}\right)}$

  Need $x^2 - 4 \neq 0 \implies (x - 2)(x + 2) \neq 0 \implies x \neq \pm 2$

  Need $\left(\dfrac{x + 1}{x^2 - 4}\right) \neq 0 \implies x + 1 \neq 0 \implies x \neq -1.$

  Hence, $\text{dom}(h) = (-\infty, -2) \cup (-2, -1) \cup (-1, 2) \cup (2, \infty)$.

Example 4.1.8. Let's rewrite the functions from the last example in the form $\frac{e(x)}{d(x)}$ for polynomials $e(x)$ and $d(x)$ in lowest terms (no common factors).

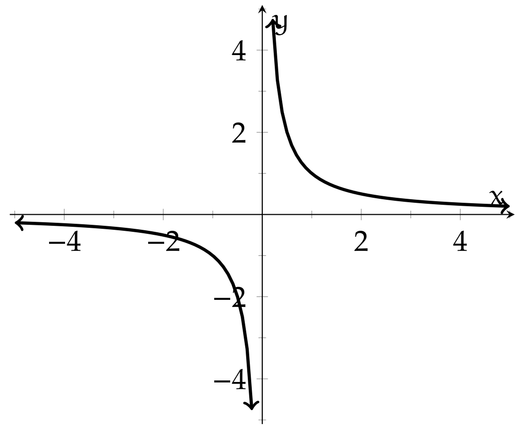
- $f(x) = \dfrac{2x - 1}{x + 1}$ is already in this form and in lowest terms.

- $g(x) = 2 - \dfrac{3}{x+1} = \dfrac{(x+1)2}{x+1} - \dfrac{3}{x+1} = \dfrac{(x+1)2-3}{x+1} = \dfrac{2x+2-3}{x+1} = \dfrac{2x-1}{x+1}.$
- $$\begin{aligned} h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} = \frac{2x^2-1-(3x-2)}{x^2-1} = \frac{2x^2-1-3x+2}{x^2-1} &= \frac{2x^2-3x+1}{x^2-1} \\ &= \frac{2x^2-2x-x+1)}{(x+1)(x-1)} \\ &= \frac{2x(x-1)-(x-1)}{(x+1)(x-1)} \\ &= \frac{(2x-1)(x-1)}{(x+1)(x-1)} \\ &= \frac{2x-1}{x+1}. \end{aligned}$$
- $r(x) = \dfrac{\left(\dfrac{2x-1}{x^2-4}\right)}{\left(\dfrac{x+1}{x^2-4}\right)} = \dfrac{2x-1}{x^2-4} \cdot \dfrac{x^2-4}{x+1} = \dfrac{(2x-1)(x^2-4)}{(x^2-4)(x+1)} = \dfrac{2x-1}{x+1}.$

So they all appear to look the same when simplified, however only $f(x) = g(x)$ (the others have different domains).

4.2 Graphs of Rational Functions

Remark 4.2.1. Consider the graph of $f(x) = \frac{1}{x}$.



We know $\text{dom}(f) = (-\infty, 0) \cup (0, \infty)$ and at $x = 0$ something peculiar happens to the graph of $f(x)$.

Definition 4.2.2. Let f be a function.

(1) A *vertical asymptote* of the graph $y = f(x)$ is a vertical line $x = c$ such that

- if $x \to c^-$ then $|f(x)| \to \infty$
- if $x \to c^+$ then $|f(x)| \to \infty$

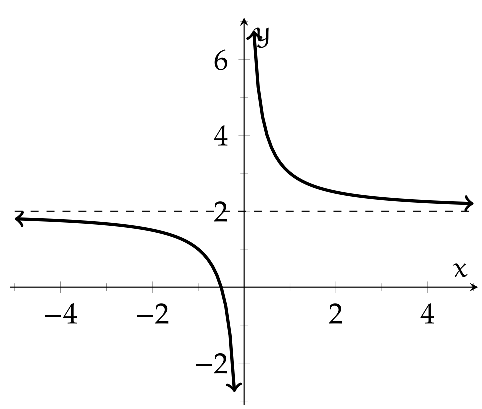
(2) A *horizontal asymptote* of the graph $y = f(x)$ is a horizontal line $y = c$ such that

- if $x \to -\infty$ then $f(x) \to c$
- if $x \to \infty$ then $f(x) \to c$.

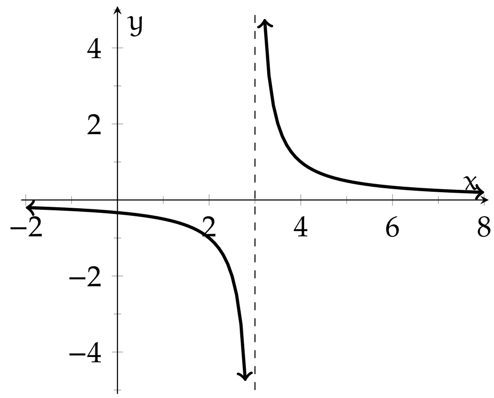
Remark 4.2.3. Observe that having a horizontal asymptote is a special instance of describing the end behavior of a function.

Example 4.2.4.

- $f(x) = \dfrac{1}{x} + 2$ has the graph of $y = \frac{1}{x}$ shifted up by 2:



- $g(x) = \dfrac{1}{x-3}$ has the graph of $y = \frac{1}{x}$ shifted right by 3:



- $h(x) = \dfrac{1+2x}{x}$

  Observe, $\dfrac{1+2x}{x} = \dfrac{1}{x} + \dfrac{2x}{x} = \dfrac{1}{x} + 2.$

  This is $f(x)$ from part a).

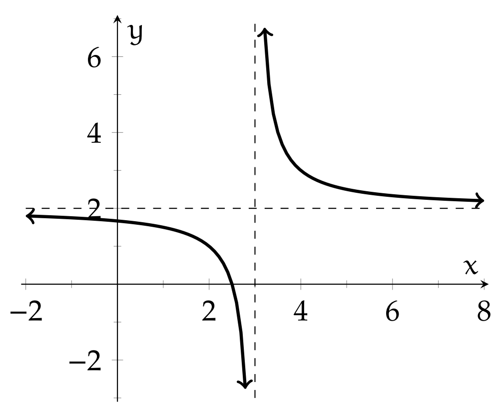
- $i(x) = \dfrac{2x-5}{x-3}$

  Observe,

$$\begin{array}{r|r} & 2 \\ \hline x-3 & 2x-5 \\ & -2x+6 \\ \hline & 1 \end{array}$$

  so $2x - 5 = 2(x-3) + 1 \quad \Longrightarrow \quad \dfrac{2x-5}{x-3} = \dfrac{1}{x-3} + 2$

  Thus, the graph $y = h(x)$ is obtained by shifting the graph $y = \frac{1}{x}$ up 2 and right 3:

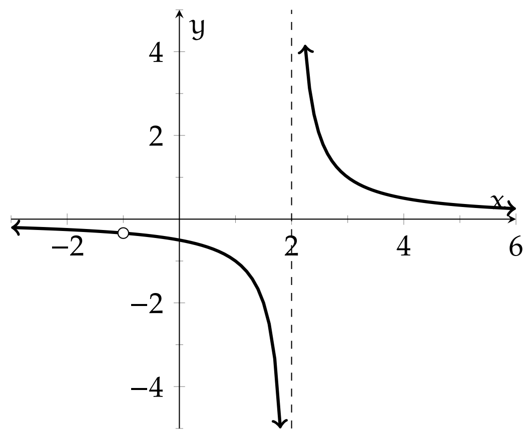


Exercise. Sketch $f(x) = \dfrac{1}{x+2} - 1$ and identify the asymptotes.

Remark 4.2.5. If $f(x)$ is obtained by transforming $\frac{1}{x}$ it is easy to find the asymptotes. However, not every rational function can be obtained this way.

Example 4.2.6. Let $f(x) = \dfrac{x+1}{x^2 - x - 2}$. Simplify: $\dfrac{x+1}{x^2-x-2} = \dfrac{1(x+1)}{(x-2)(x+1)} = \dfrac{1}{x-2}$.

Thus, the graph of $f(x)$ is obtained by transforming $\frac{1}{x}$ right by 2. However note that $\text{dom}(f) = (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$, so $y = f(x)$ has a hole on the vertical line $x = 2$:



We say the graph of $f(x)$ has a *hole* at $(2, -1)$.

Definition 4.2.7. Let $f(x)$ be a function. A point (a, b) is a *hole* of the graph $y = f(x)$ if

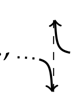
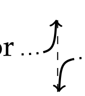
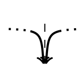
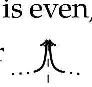
(1) $a \notin \text{dom}(f)$

(2) The graph $y = f(x)$ goes through (a, b).

Remark 4.2.8. Determining the holes and vertical asymptotes are crucial to sketch a graph. They can be found by factoring the enumerator and denominator of a rational function!

Fact 4.2.9. Suppose $f(x) = \frac{e(x)}{d(x)}$ is a rational function and c is a zero of multiplicity m of $d(x)$. (Equivalently, $(x-c)^m$ is a factor of $d(x)$)
If c is a zero of $e(x)$, let n be its multiplicity. If c is a not a zero of $e(x)$, let $n = 0$. (Equivalently, $(x-c)^n$ is a factor of $e(x)$ or $(x-c)^0 = 1$ is a factor of $e(x)$)

(1) If $m > n$, then $x = c$ is a vertical asymptote.

- If $m - n$ is odd, then $f(x)$ flips signs over the vertical asymptote. E.g.,  or 
- If $m - n$ is even, then $f(x)$ has the same sign on both sides of the vertical asymptote. E.g.,  or 

(2) If $m \leqslant n$ then the graph has a hole on the line $x = c$. In particular, the hole is $(c, f'(c))$ where $f'(x)$ is obtained by removing the common factors from $f(x)$.

Example 4.2.10.

- $f(x) = \dfrac{2x}{x^2 - 9} = \dfrac{2x}{(x-3)(x+3)}$.
  - $(x - 3)$ is only a factor of the denominator, so $x = 3$ is a vertical asymptote.
  - $(x + 3)$ is only a factor of the denominator, so $x = -3$ is a vertical asymptote.
- $g(x) = \dfrac{x^2 - x - 6}{x^2 - 9} = \dfrac{(x+2)(x-3)}{(x-3)(x+3)}$
  - $(x - 3)$ is a common factor, so there is a hole on the line $x = 3$. Cancelling factors gives denominator $f'(x) = \frac{(x+2)}{(x+3)}$, so the hole is $(3, \frac{5}{6})$.
  - $(x + 3)$ is only a factor of the denominator, so $x = -3$ is a vertical asymptote.
- $h(x) = \dfrac{x^2 - x - 6}{x^2 + 9} = \dfrac{(x-3)(x+2)}{x^2 + 9}$.
  - Note that $d(x) = x^2 + 9$ has no zeros, and hence no factors. Thus, $y = f(x)$ has no holes and no vertical asymptotes.
- $r(x) = \dfrac{x^2 - x - 6}{x^2 + 4x + 4} = \dfrac{x^2 - x - 6}{x^2 + 4x + 4} = \dfrac{(x+2)(x-3)}{(x+2)(x+2)} = \dfrac{(x+2)(x-3)}{(x+2)^2}$
  - $(x - 2)^2$ is only a factor of the denominator, so $x = 2$ is a vertical asymptote.
- $p(x) = \dfrac{(x-4)(x-3)(x-2)(x+2)}{(x-5)(x-3)(x+2)(x+3)}$.
  - $(x - 5)$ is only a factor of the denominator, so $x = 5$ is a vertical asymptote.
  - $(x - 3)$ is a common factor and $f'(3) = \frac{(3-4)(3-2)}{(3-5)(3+3)} = \frac{1}{12}$, so there is a hole at $(3, \frac{1}{12})$
  - $(x + 2)$ is a common factor and $f'(-2) = \frac{(-2-4)(-2-2)}{(-2-5)(-2+3)} = -\frac{24}{7}$, so there is a hole at $(-2, -\frac{24}{7})$.

Exercise. Find the holes and vertical asymptotes of $f(x) = \dfrac{(x-4)(x-2)(x+1)}{(x-2)(x-1)(x+3)}$.

Remark 4.2.11. We can now find vertical asymptotes and holes of factored rational functions. We only need one more ingredient to make rough sketches of rational functions, the ability to find the end behavior, i.e., determining whether/where the function has a horizontal asymptote.

Fact 4.2.12. Suppose $f(x) = \frac{e(x)}{d(x)}$ is a rational function, where $e(x)$ and $d(x)$ have the leading terms ax^n and bx^m, respectively.

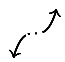
(1) The end behavior of $f(x)$ is the same as the end behavior of $y = \dfrac{ax^n}{bx^m}$ $(= \frac{a}{b}x^{n-m})$.

(2) If $n = m$, then $y = \frac{a}{b}$ is the horizontal asymptote of the graph of $y = f(x)$.

(3) If $n < m$, then $y = 0$ is the horizontal asymptote of the graph of $y = f(x)$.

(4) If $m > n$, then the graph of f has no horizontal asymptotes.

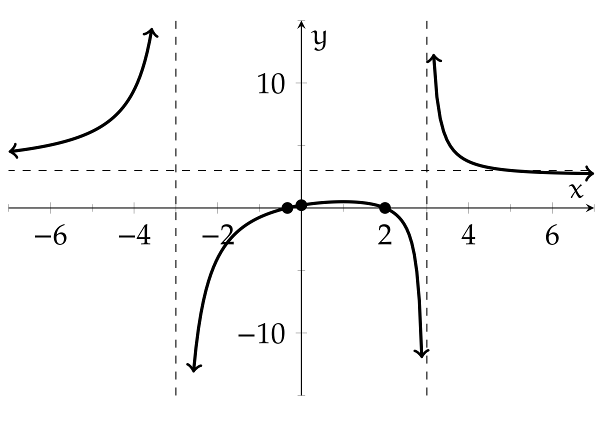
Example 4.2.13.

- $f(x) = \dfrac{5x}{x^2+1}$
  - $f(x)$ has the same end behavior as $y = \frac{5x}{x^2} = \frac{5}{x}$.
  - The horizontal asymptote is $y = 0$.
- $g(x) = \dfrac{x^2-4}{x+1}$
  - $f(x)$ has the same end behavior as $y = \frac{x^2}{x} = x$
  - The end behavior of $f(x)$ is .
  - There is no horizontal asymptote.
- $h(x) = \dfrac{6x^3-3x+1}{5-2x^3}$
  - $f(x)$ has the same end behavior as $y = \frac{6x^3}{-2x^3} = -3$
  - The horizontal asymptote is $y = -3$.

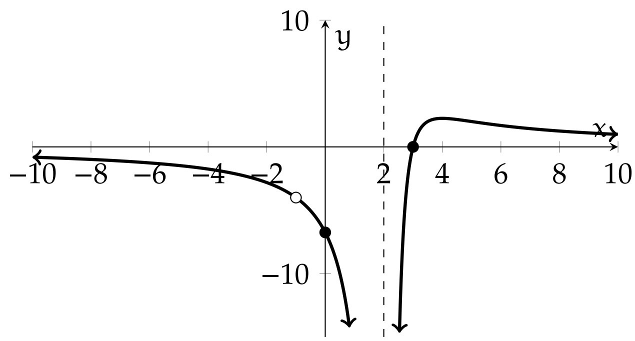
We are now ready to sketch rational functions.

Example 4.2.14.

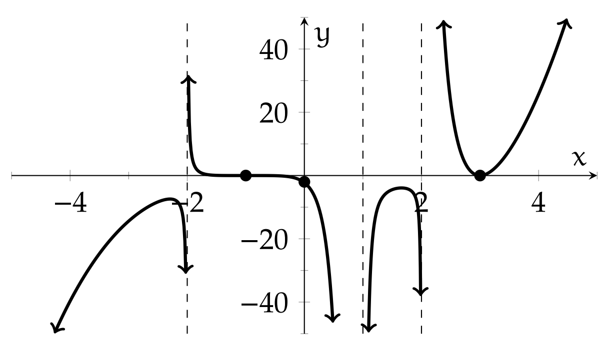
- $f(x) = \dfrac{3x^2-5x-2}{x^2-9} = \dfrac{3x^2-6x+1x-2}{(x-3)(x+3)} = \dfrac{3x(x-2)+1(x-2)}{(x-3)(x+3)} = \dfrac{(3x+1)(x-2)}{(x-3)(x+3)}$.
  - Vertical asymptote at $x = -3$ (flips signs)
  - Vertical asymptote at $x = 3$ (flips signs)
  - $\dfrac{3x^2}{x^2} = 3 \implies y = 3$ is the horizontal asymptote.
  - x-intercepts are $(-\frac{1}{3}, 0)$ and $(2, 0)$ (both cross x-axis)
  - y-intercept is $(0, \frac{2}{9})$



- $g(x) = \dfrac{9(x-3)^2(x+1)}{(x-2)^2(x+1)}$
  - Vertical asymptote at $x = 2$ (same sign)
  - Hole at $(-1, -4)$
  - $\dfrac{9x^2}{x^3} = \dfrac{9}{x} \implies y = 0$ is the horizontal asymptote
  - x-intercept is $(3, 0)$ (crosses x-axis)
  - y-intercept is $(0, -\frac{27}{4})$



- $h(x) = \dfrac{8(x-3)^2(x+1)^5}{9(x+2)(x-2)(x-1)^2}$
  - Vertical asymptote at $x = -2$ (flips signs)
  - Vertical asymptote at $x = 2$ (flips signs)
  - Vertical asymptote at $x = 1$ (same sign)
  - $\dfrac{x^7}{x^4} = x^3 \implies$ end behavior is $\swarrow \cdots \nearrow$
  - x-intercept at $(3, 0)$ (bounces off x-axis)
  - x-intercept at $(-1, 0)$ (crosses through x-axis)
  - y-intercept at $(0, -2)$



Exercise. Sketch the graph of $f(x) = \dfrac{2(x-2)^2(x+1)}{(x-1)^2(x+1)}$.

4.3 Rational ~~In~~equalities and Applications

Definition 4.3.1. Suppose x, y are variables. We say

(1) y *varies directly with* x if there is some $k \in (-\infty, \infty)$ such that $y = kx$.

(2) y *varies inversely with* x if there is some $k \in (-\infty, \infty)$ such that $y = \frac{k}{x}$.

The value k is called the *constant of proportionality*.

Example 4.3.2.

- $y = 2x \implies y$ varies directly with x and the constant of proportionality $k = 2$.
- $P = \frac{1}{5}t \implies P$ varies directly with t.
- $m = \frac{1}{s} \implies m$ varies inversely with s.
- $d = \frac{1}{2}t^2 \implies d$ varies directly with t^2.
- $xy = 4 \implies x$ varies inversely with y.
- $A = 42ks \implies A$ varies directly with k and s.
- Let $c \in (-\infty, \infty)$ and suppose $x = 2c$. Then x does not vary directly with c.

Example 4.3.3.

- "At fixed pressure, the temperature T of an ideal gas is directly proportional to its volume V."
  $\implies T = kV$.
- "The frequency of a wave f is inversely proportional to the wavelength of the wave λ".
  $\implies f = \dfrac{k}{\lambda}$.
- "The force F exerted on a spring is directly proportional the extension x of the spring."
  $\implies F = kx$.
- "The volume V of a cone varies with the height h and the square of the radius r of the base."
  $\implies V = khr^2$
- "The current I directly proportional to the voltage V and inversely proportional to the resistance R."
  $\implies I = \dfrac{kV}{R}$.
- "The foo f varies directly with the square root of the bar b and inversely with the absolute value of the foobar F".
  $\implies f = \dfrac{k\sqrt{b}}{|F|}$

Example 4.3.4.

- Suppose that y varies directly with x. If $x = 11$ then $y = 42$.

  We know $y = kx$ and we want to find k. Plug in $x = 11$ and $y = 42$.

  $$42 = 11k \quad \Longrightarrow \quad k = \frac{42}{11}.$$

- Suppose z is directly proportional to x and inversely proportional to z. If $y = 4$ when $x = 2$ and $z = 5$.

  We have $y = \frac{kx}{z}$, so plugging in $y = 4$, $x = 2$, and $z = 5$ we get

  $$4 = \frac{2k}{5} \quad \Longrightarrow \quad k = \frac{4 \cdot 5}{2} = \frac{20}{2} = 10.$$

Example 4.3.5. The velocity v of a monkey rocket varies directly to the time t after the launch. If after 2 seconds, the velocity of the rocket is 64 feet per second, what is the velocity after 5 seconds?

First, we set up the mathematical notation using variation. We get $v = kt$. Now plugging in $t = 2$ and $v = 64$ we get

$$64 = 2k \quad \Longrightarrow \quad k = 32$$

Thus, $v = 32t$. Then solving v for $t = 5$ we have $v = 32 \cdot 5 = 160$ feet per second.

Exercise. Use the given information to find the (i) the constant of proportionality, and (ii) the wanted value y.

a) y varies directly with x.
If $x = 2$, then $y = 10$.
Wanted: y when $x = 5$.

b) y is inversely proportional to x.
If $x = 5$, then $y = 10$.
Wanted: y when $x = 25$.

c) y varies jointly with x and z
If $x = 4$ and $z = 2$, then $y = 16$.
Wanted: y when $x = 3$ and $z = 3$.

d) y is directly proportional to the cube of x.
If $x = 2$, then $y = 4$.
Wanted: y when $x = 3$.

Remark 4.3.6. To simplify $\frac{4}{5} + \frac{3}{8}$ we need to find a common denominator. This can easily be done:

$$\frac{4}{5} + \frac{3}{8} = \frac{4(8)}{5(8)} + \frac{3(5)}{8(5)} = \frac{32}{40} + \frac{15}{40} = \frac{47}{80}$$

Similarly, we find common denominators between rational expressions:

$$\frac{2x+1}{x+1} + \frac{3}{4x+2} = \frac{(2x+1)(4x+2)}{(x+1)(4x+2)} + \frac{3(x+1)}{(x+1)(4x+2)} = \frac{6x^2+8x+2}{4x^2+6x+2} + \frac{3x+3}{4x^2+6x+2} = \frac{6x^2+11x+5}{4x^2+6x+2}$$

Exercise. Simplify the following.

a) $\dfrac{1}{x+3} + \dfrac{2}{x-1}$

b) $\dfrac{2}{x-2} - \dfrac{1}{x}$

c) $\dfrac{1}{x} + \dfrac{1}{x+2} + \dfrac{1}{x-2}$

Example 4.3.7.

- $$\begin{aligned} \frac{x}{5x+4} = 3 \quad &\implies \quad x = 3(5x+4) \\ &\implies \quad x = 15x + 12 \\ &\implies \quad -14x = 12 \\ &\implies \quad x = -\frac{12}{14} = -\frac{6}{7} \end{aligned}$$

- $$\begin{aligned} \frac{1}{x+2} + \frac{3}{x} = -2 \quad &\implies \quad \frac{x}{x^2+2x} + \frac{3x+6}{x^2+2x} = -2 \\ &\implies \quad \frac{4x+6}{x^2+2x} = -2 \\ &\implies \quad 4x+6 = -2x^2 - 4x \\ &\implies \quad 2x^2 + 8x + 6 = 0 \\ &\implies \quad 2x^2 + 6x + 2x + 3 = 0 \\ &\implies \quad 2x(x+3) + 2(x+3) = 0 \\ &\implies \quad (2x+2)(x+3) = 0 \end{aligned}$$

  $\implies x = -1$ or $x = -3$

Remark 4.3.8. When solving rational equations you might cancel out a factor or multiply by a denominator. This might introduce "solutions" that do not solve the original equation.

Example 4.3.9.

- $$\begin{aligned} \frac{x^3-2x+1}{x-1} = \frac{x-2}{2} \quad &\implies \quad 2(x^3-2x+1) = (x-2)(x-1) \\ &\implies \quad 2x^3 - 4x + 2 = x^2 - x - 2x + 2 \\ &\implies \quad 2x^3 - x^2 - 3x = 0 \\ &\implies \quad x(2x^2 - x - 3) = 0 \\ &\implies \quad x(2x+1)(x-1) = 0 \end{aligned}$$

  $\implies x = 0$ or $x = -\frac{1}{2}$ or $x = 1$

  However, plugging in $x = 1$ gives division by $0 \implies x = 1$ is not a solution.

- $$\begin{aligned} \frac{1}{x+3} + \frac{1}{x-3} = \frac{x^2-3}{x^2-9} \quad &\implies \quad \frac{x-3+x+3}{x^2-9} = \frac{x^2-3}{x^2-9} \\ &\implies \quad \frac{2x}{x^2-9} = \frac{x^2-3}{x^2-9} \\ &\implies \quad 2x = x^2 - 3 \\ &\implies \quad -x^2 + 2x + 3 = 0 \\ &\implies \quad x^2 - 2x - 3 = 0 \\ &\implies \quad (x+1)(x-3) = 0 \end{aligned}$$

- $\dfrac{19+x}{|x+3|} = 3 \implies 19 + x = 3|x+3|$

$$\begin{aligned}
&\implies 19 + x = 3(x+3) \quad \text{or} \quad 19 + x = -3(x+3) \\
&\implies 19 + x = 3x + 9 \quad \text{or} \quad 19 + x = -3x - 9 \\
&\implies -2x = -10 \quad \text{or} \quad 4x = -28 \\
&\implies x = 5 \quad \text{or} \quad x = -7
\end{aligned}$$

Exercise. Solve the following equations.

a) $\dfrac{x^2 - x}{2} = 1$ b) $\dfrac{1}{x} = \dfrac{5}{2}$ c) $\dfrac{x+3}{x^3 + x + 3} = 0$

d) $\dfrac{3}{2x^2 + 5x} = 1$ e) $\dfrac{1}{x+8} - \dfrac{1}{3x+2} = 0$ f) $\dfrac{4x+8}{x-2} = x + 2$

Example 4.3.10. The population of super intelligent monkeys on Mars is modeled by the function

$$P(t) = \frac{500t}{2t + 100},$$

where $t = 0$ represents the year 1885.

- How many monkeys are there in 1935?

  $1935 - 1885 = 50$ so we need to evaluate $p(50) = \dfrac{500(50)}{2(50) + 100} = \dfrac{25000}{200} = 125$ monkeys

- When are there 200 super intelligent monkeys on Mars?

  We have to solve the equation $P(t) = 200$.

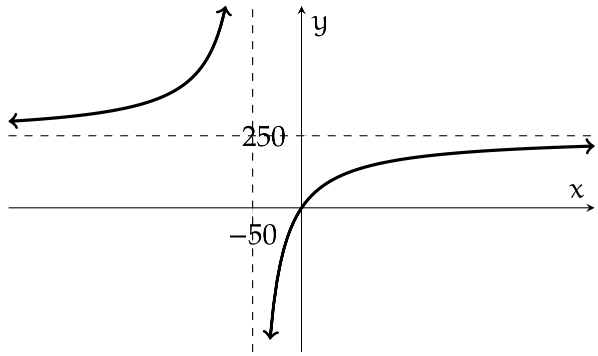
$$\begin{aligned}
\frac{500t}{2t+100} = 200 &\implies 500t = 200(2t + 100) \\
&\implies 500t = 400t + 20000 \\
&\implies 100t = 20000 \\
&\implies t = 200
\end{aligned}$$

  $t = 200$ represents the year $1885 + 200 = 2285$, so there are 200 monkeys in 2285.

- Does the population reach a maximum?

  We can sketch $y = P(t)$:

  - Vertical asymptote at $t = -50$
  - End behavior same as $y = \frac{500t}{2t} = 250 \implies$ horizontal asymptote at $y = 250$.
  - x-intercept $(0, 0)$, also the y-intercept.



The population does never go above 250.

5.1 Function Composition

Exercise. Let $f(x)$ and $g(x)$ be functions with $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Unfold the definitions.

a) $(f+g)(x) =$ b) $(f-g)(x) =$ c) $(fg)(x) =$ d) $\left(\frac{f}{g}\right)(x) =$

Definition 5.1.1. Suppose f and g are functions. The *composite* of g with f, denoted $g \circ f$ is defined by the formula

$$(g \circ f)(x) = g(f(x)),$$

provided $x \in \mathrm{dom}(f)$ and $f(x) \in \mathrm{dom}(g)$.

Remark 5.1.2. We read $g \circ f$ as "g composed with f," or "g of f."

Example 5.1.3.

- Let $f(x) = x^2$ and $g(x) = x + 1$.
  - $(g \circ f)(2) = g(f(2)) = g(2^2) = g(4) = 4 + 1 = 5$
  - $(f \circ g)(2) = f(g(2)) = f(2+1) = f(3) = 3^2 = 9$
  - $(f \circ g \circ f)(1) = f(g(f(1))) = f(g(1^2)) = f(g(1)) = f(1+1) = f(2) = 2^2 = 4$
  - $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$
  - $\mathrm{dom}(g \circ f) = (-\infty, \infty)$
  - $(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$
  - $\mathrm{dom}(f \circ g) = (-\infty, \infty)$
- Let $f(x) = |x|$ and $g(x) = \sqrt{x}$.
  - $(f \circ g)(9) = f(g(9)) = f(\sqrt{9}) = f(3) = |3| = 3$
  - $(f \circ g)(-9) = f(g(-9)) = f(\sqrt{-9}) = ? =$ undefined because $-9 \notin \mathrm{dom}(g)$
  - $(g \circ f)(-5) = g(f(-5)) = g(|-5|) = g(5) = \sqrt{5}$
  - $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = |\sqrt{x}| = \sqrt{x}$
  - $\mathrm{dom}(f \circ g) = [0, \infty)$
  - $(g \circ f)(x) = g(f(x)) = f(|x|) = \sqrt{|x|}$
  - $\mathrm{dom}(g \circ f) = (-\infty, \infty)$ because $|x| \geqslant 0$ is true for all x.
- Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{2}{x+3}$.
  - $(g \circ f)(3) = g(f(3)) = g\left(\frac{1}{3^2}\right) = g\left(\frac{1}{9}\right) = \frac{2}{\frac{1}{9}+3} = \frac{2}{\frac{1}{9}+\frac{27}{9}} = \frac{2}{\frac{28}{9}} = 2 \cdot \frac{9}{28} = \frac{18}{28} = \frac{9}{14}$
  - $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x^2}\right) = \frac{2}{\frac{1}{x^2}+3} = \frac{2}{\frac{1}{x^2}+\frac{3x^2}{x^2}} = \frac{2}{1+3x^2}x^2 = 2 \cdot \frac{x^2}{1+3x^2} = \frac{2x^2}{1+3x^2}$
  - $\mathrm{dom}(g \circ f) = (-\infty, 0) \cup (0, \infty)$

Remark 5.1.4. To find the domain unfold the functions and find the domain as usual. Do not simplify and then find the domain!

Exercise. Let $f(x) = |x|$ and $g(x) = -x$. Evaluate the following expressions.

a) $f(-2)$ b) $g(-2)$ c) $(g \circ f)(-2)$ d) $(f \circ g)(-2)$

e) $(f + g)(-2)$ f) $(g - f)(-2)$ g) $(gf)(-2)$ h) $\left(\frac{f}{g}\right)(-2)$

i) $g(1)$ j) $(g \circ g)(1)$ k) $(g \circ g \circ g)(1)$ l) $(f \circ g \circ g \circ g)(1)$

Remark 5.1.5. We now know how to compute the composition of two functions. It can be useful go the other way; to write a function as the composition of simpler functions.

Example 5.1.6.

- $f(x) = |3x|$
  - Want $f_1(x)$, $f_2(x)$ such that $f(x) = (f_2 \circ f_1)(x)$
  - $f_1(x) = 3x$
  - $f_2(x) = |x|$
  - $(f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(3x) = |3x| = f(x)$
- $g(x) = \dfrac{2}{x^2 + 1}$
  - $g_1(x) = x^2$
  - $g_2(x) = x + 1$
  - $g_3(x) = \dfrac{2}{x}$
  - $(g_3 \circ g_2 \circ g_1)(x) = g_3(g_2(g_1(x))) = g_3(g_2(x^2)) = g_3(x^2 + 1) = \dfrac{2}{x^2 + 1} = g(x)$
- $h(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$
  - $h_1(x) = \sqrt{x}$
  - $h_2(x) = \dfrac{x + 1}{x - 1}$
  - $(h_2 \circ h_1)(x) = h_2(h_1(x)) = h_2(\sqrt{x}) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1} = h(x)$

Exercise. Write the given function as a composition of two or more (non-identity) functions.

a) $f(x) = 2x + 3$ b) $g(x) = \sqrt{x^2 - 4}$ c) $h(x) = |x|^2 + |x|$ d) $p(x) = \dfrac{1 - |x + 4|}{|x + 4| + 2}$

Example 5.1.7. Let $f(x)$ be a function and $i(x) = x$ the identity function.

- $(i \circ f)(x) = i(f(x)) = f(x)$.
- $(f \circ i)(x) = f(i(x)) = f(x)$.

Compare this to multiplying a number by 1.

5.2 Inverse functions

Example 5.2.1. Let $f(x) = 3x + 4$. Recall, we can describe functions in terms of "recipes." That is, f performs the following steps in order.

1. Multiply by 3
2. Add 4

We can define a new function by performing the inverses of the operations in reverse order, i.e., the steps

1. Subtract 4
2. Divide by 3

This gives the function $g(x) = \frac{x-4}{3}$.

You might expect that these functions are each others "inverse." Indeed, observe the following.

- $(g \circ f)(2) = g(f(2)) = g(3(2) + 4) = g(10) = \frac{10-4}{3} = 2$
- $(g \circ f)(-3) = g(f(-3)) = g(3(-3) + 4) = g(-5) = \frac{-5-4}{3} - 3$
- $(f \circ g)(4) = f(g(4)) = f\left(\frac{4-4}{3}\right) = f(0) = 3(0) + 4 = 4$
- $(f \circ g)(x) = f(g(x)) = 3g(x) + 4 = 3\left(\frac{x-4}{3}\right) + 4 = \frac{3(x-4)}{3} + 4 = x - 4 + 4 = x$
- $(g \circ f)(x) = g(f(x)) = \frac{f(x)-4}{3} = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x.$

We have $(f \circ g)(x) = x$ and $(g \circ f)(x)$ for all x. This means that whatever f does is undone by g and vice versa, we will call g the *inverse* of f.

Definition 5.2.2. A function $f(x)$ is called *invertible* if there exists a function $g(x)$ such that

- $(f \circ g)(x) = x$ for all $x \in \text{dom}(g)$.
- $(g \circ f)(x) = x$ for all $x \in \text{dom}(f)$

$g(x)$ is called the *inverse* of $f(x)$, and we commonly write $f^{-1}(x)$ for $g(x)$.

Example 5.2.3.

- Let $f(x) = x + 4$ and $g(x) = x - 4$. Then
  - Let $x \in \text{dom}(g) = (-\infty, \infty)$. Then $(f \circ g)(x) = f(g(x)) = f(x-4) = (x-4) + 4 = x$
  - Let $x \in \text{dom}(f) = (-\infty, \infty)$. Then $(g \circ f)(x) = g(f(x)) = g(x+4) = (x+4) - 4 = x$.

  g is the inverse of f.

- Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. We want to know if $g(x)$ is the inverse of $f(x)$.
  - Let $x \in \text{dom}(g) = [0, \infty)$. Then $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$.
  - Let $x \in \text{dom}(f) = (-\infty, \infty)$. Then $(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x| \neq x$.

  $g(x)$ is not the inverse of $f(x)$.

Exercise. Determine whether $f(x) = 4 - 2x$ and $g(x) = 2 - \frac{1}{2}x$ are each others inverse.

Fact 5.2.4. Let $f(x)$ be a invertible function, $f^{-1}(x)$ its inverse, and $a, b \in (-\infty, \infty)$. The following are equivalent

(1) $f(a) = b$

(2) $f^{-1}(b) = a$

(3) (a, b) is on the graph $y = f(x)$.

(4) (b, a) is on the graph $y = f^{-1}(x)$.

Example 5.2.5. Suppose $f(x)$ is an invertible function, $f^{-1}(x)$ is its inverse, and $c \in (-\infty, \infty)$.

- $(c, 0)$ is an x-intercept of $f \iff f(c) = 0 \iff f^{-1}(0) = c \iff (0, c)$ is the y-intercept of f^{-1}.

- $f(x)$ cannot have two different x-intercepts. Suppose $(c, 0)$ and $(d, 0)$ are x-intercepts of $f(x)$. Then $(0, c)$ and $(0, d)$ are y-intercepts of $f^{-1}(x)$. But a function cannot have different y-intercepts, so $c = d$.

- $f(x) = c$ cannot have more than one solution for the same reason.

Remark 5.2.6. $f(x) = x^2$ has no inverse because there are two solutions to $f(x) = 4$.

Definition 5.2.7. A function f is *one-to-one* if $a, b \in \text{dom}(f)$ with $f(a) = f(a)$ implies $a = b$.

Remark 5.2.8. A function f is one-to-one if and only if $a \neq b$ implies $f(a) \neq f(b)$.

Example 5.2.9.

- $f(x) = x^2$ is not one-to-one; $f(2) = 4 = f(-2)$ but $2 \neq -2$.

- $i(x) = x$ is one-to-one; $i(a) = i(b)$ implies $a = b$.

- $A(x) = |x|$ is not one-to-one; $A(1) = 1 = A(-1)$ but $1 \neq -1$.

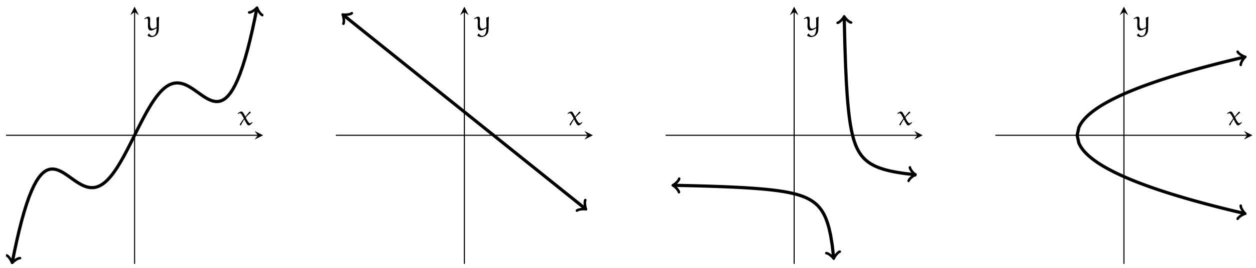
Fact 5.2.10 (Horizontal line test)**.** Suppose $f(x)$ is a function. The following are equivalent.

(1) $f(x)$ is invertible.

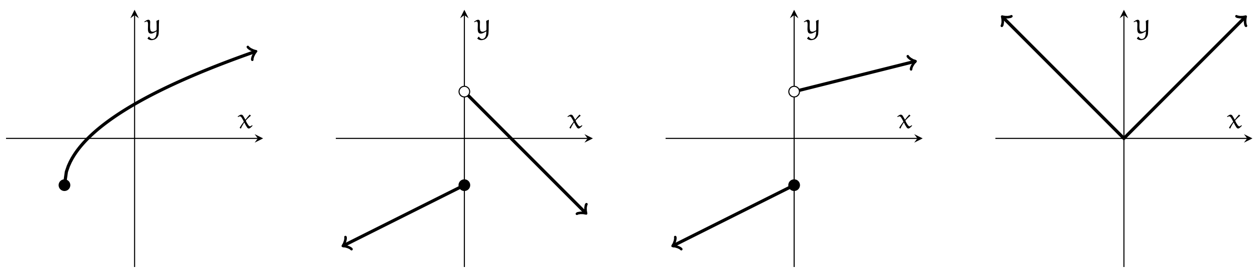
(2) $f(x)$ is one-to-one.

(3) No horizontal line intersects the graph $y = f(x)$ more than once.

Example 5.2.11. Which of the following graphs correspond to invertible functions?



Exercise. Which of the following graphs correspond to one-to-one functions?



Fact 5.2.12. Suppose $f(x)$ is invertible.

(1) The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

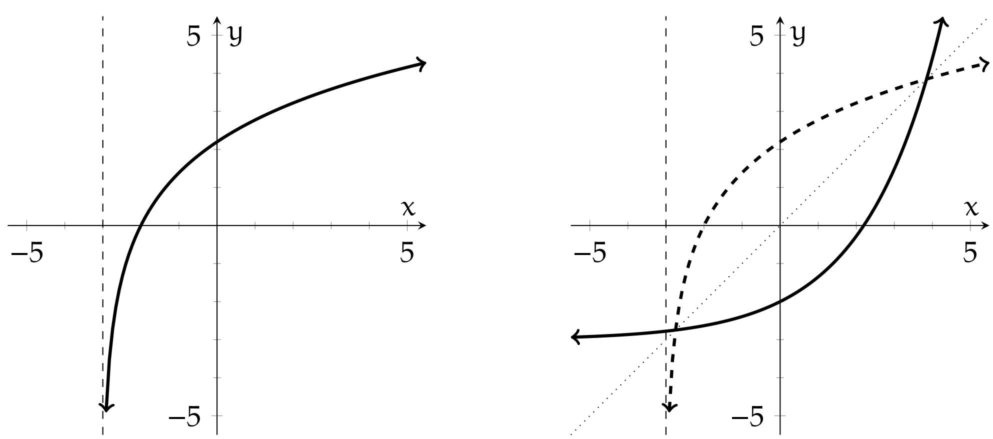
(2) $\text{ran}(f) = \text{dom}(f^{-1})$

(3) $\text{dom}(f) = \text{ran}(f^{-1})$

(4) $f(a) = b$ iff $f^{-1}(b) = a$

(5) (a, b) is on the graph $y = f(x)$ iff (b, a) is on the graph of $y = f^{-1}(x)$

Example 5.2.13. Let $f(x)$ be a function with the graph below on the left.



How many solutions are there for $f^{-1}(x) = -4$? We can sketch $y = f^{-1}(x)$ to answer this question.

Example 5.2.14. Show what happens when we reflect the graph of $f(x) = x^2$ across the line $y = x$.

Fact 5.2.15. The inverse of a function is unique (there is only one reflection across the line $x = y$).

Remark 5.2.16. Steps for finding the inverse of an invertible function.

I. Write $y = f(x)$.

II. Interchange x and y.

III. Solve $x = f(y)$ for $y \in \text{dom}(f)$.

IV. Write $f^{-1}(x) =$ (the solution for y)

Example 5.2.17.

- Let $f(x) = \dfrac{1-2x}{5}$.
  I. $y = \dfrac{1-2x}{5}$
  II. $x = \dfrac{1-2y}{5}$
  III.
  $$\begin{aligned} x &= \frac{1-2y}{5} \\ 5x &= 1-2y \\ 5x-1 &= -2y \\ \frac{5x-1}{-2} &= y \end{aligned}$$
  IV. $f^{-1}(x) = \dfrac{5x-1}{-2}$

- Let $h(x) = \sqrt{x+1} - 4$.
  I. $y = \sqrt{x+1} - 4$
  II. $x = \sqrt{y+1} - 4$
  III.
  $$\begin{aligned} x &= \sqrt{y+1} - 4 \\ x+4 &= \sqrt{y+1} \\ (x+4)^2 &= y+1 \\ (x+4)^2 - 1 &= y \end{aligned}$$
  IV. $h^{-1}(x) = (x+4)^2 - 1$
  But $\text{dom}(h^{-1}) = \text{ran}(h) = [-4, \infty)$, so $h(x)$ is not the inverse of the polynomial $p(x) = (x+4)^2 - 1$

- Let $g(x) = \dfrac{1}{x}$
  I. $y = \dfrac{1}{x}$
  II. $x = \dfrac{1}{y}$
  III.
  $$\begin{aligned} x* &= \frac{1}{y} \\ xy &= 1 \\ y &= \frac{1}{x} \end{aligned}$$
  IV. $g^{-1}(x) = \dfrac{1}{x} = g(x)$.

- Let $p(x) = (x+4)^2 - 1$.
  I. $y = (x+4)^2 - 1$
  II. $x = (y+4)^2 - 1$
  III.
  $$\begin{aligned} x &= (y+4)^2 - 1 \\ x+1 &= (y+4)^2 \\ \pm\sqrt{x+1} &= y+4 \\ \pm\sqrt{x+1} - 4 &= y \end{aligned}$$
  The method can't work if the function is not invertible.

Exercise. Find the inverse of the following one-to-one functions.

a) $f(x) = 6x - 2$ b) $g(x) = \frac{1}{3}x + 7$ c) $h(x) = \dfrac{1-2x}{5}$ d) $p(x) = \dfrac{3}{4-x}$

5.3 Radical functions

Fact 5.3.1. Let $a, b \in (-\infty, \infty)$ and m, n integers. Then

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{m \cdot n} \qquad a^0 = 1$$

$$(ab)^m = a^m b^m \qquad a^{-n} = \frac{1}{a^n} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad a^1 = a$$

provided the expressions are well defined (e.g., no division by 0 etc.).

Example 5.3.2 (Simplify the expression)**.**

- $\frac{(4^5)^{25}}{4^{123}} = \frac{4^{125}}{4^{123}} = 4^{125-123} = 4^2 = 16$
- $(x^2x^3)^{-3} = (x^5)^{-3} = \frac{1}{(x^5)^3} = \frac{1}{x^{15}}$
- $1^0 = 1$ and $0^1 = 0$
- $((-3)^2)^{\frac{1}{2}}$ – the fact does not apply here!
- $\frac{2x^2y^{-2}}{4x^{-3}y} = \frac{2}{4} \cdot \frac{x^2}{x^{-3}} \cdot \frac{y^{-2}}{y} = \frac{1}{2} \cdot x^{2-(-3)} \cdot y^{-2-1} = \frac{1}{2} \cdot x^5 \cdot y^{-3} = \frac{1}{2}x^5 \cdot \frac{1}{y^3} = \frac{x^5}{2y^3}$

Exercise. Evaluate and simplify the following expressions.

a) $51^7 \cdot 51^{-5} \cdot 51^{-2}$

b) $\frac{x^{-2}y^3}{x^4y^2}$

Fact 5.3.3. Let n be a positive integer.

- If n is odd then the polynomial $f(x) = x^n$ $(\text{dom}(f) = (-\infty, \infty))$ is invertible.
- If n is even then the function $f(x) = x^n$ with $\text{dom}(f) = [0, \infty)$ is invertible.

Definition 5.3.4. Let n be a positive integer.

- If n is odd and then $\sqrt[n]{x}$ is the inverse of the polynomial $f(x) = x^n$.
- If n is even and then $\sqrt[n]{x}$ is the inverse of the function $f(x) = x^n$ with $\text{dom}(f) = [0, \infty)$.

These are called *radical functions*.

Example 5.3.5.

- $\sqrt[1]{4} = 4$ because $4^1 = 4$
- $\sqrt[2]{9} = 3$ because $3^2 = 9$.
- $\sqrt[3]{8} = 2$ because $2^3 = 8$
- $\sqrt[4]{1} = 1$ because $1^4 = 1$
- $\sqrt[1]{-4} = -4$ because $(-4)^1 = 4$
- $\sqrt[2]{-9}$ is undefined because $-25 \notin [0, \infty)$.
- $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$
- $\sqrt[4]{-1}$ is undefined because $-1 \notin [0, \infty)$.

Remark 5.3.6.

(1) $\sqrt[1]{x} = x$ for all $x \in (-\infty, \infty)$.

(2) We write $\sqrt{x}$ for $\sqrt[2]{x}$.

(3) Let $f(x) = \sqrt[n]{x}$.

- If n is odd then $\text{dom}(f) = (-\infty, \infty)$ and $\text{ran}(f) = (-\infty, \infty)$.
- If n is even then $\text{dom}(f) = [0, \infty)$ and $\text{ran}(f) = [0, \infty)$.

Example 5.3.7.

- Let $f(x) = 4 + \sqrt[4]{3x - 12}$. Then $3x - 12 \geqslant 0 \iff 3x \geqslant 12 \iff x \geqslant 4$. Hence, $\text{dom}(f) = [4, \infty)$.
- Let $g(x) = 4 + \sqrt[3]{3x - 12}$. Then $\text{dom}(g) = (-\infty, \infty)$.

Fact 5.3.8. Let $a, b \in (-\infty, \infty)$ and m and n be positive integers. Then

- $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- If n is odd then $\sqrt[n]{a^n} = a$
- If n is even then $\sqrt[n]{a^n} = |a|$

provided all expressions are defined.

Example 5.3.9 (Evaluate)**.**

- $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$
- $(\sqrt[4]{3})^2 = \sqrt[4]{3^2} = \sqrt[4]{9}$.
- $(\sqrt[4]{-3})^2$ is undefined
- $\sqrt[3]{-4} \cdot \sqrt[3]{16} = \sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$
- $\sqrt[4]{-2} \cdot \sqrt[4]{-8}$ is undefined
- $\sqrt[4]{x} \cdot \sqrt[4]{x^3} = \sqrt[4]{x^4} = |x| = x$ (assuming $x \geqslant 0$)

Definition 5.3.10. Let $x \in (-\infty, \infty)$ and m, n integers with $n > 0$. We define $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$ provided the right hand side is defined.

Fact 5.3.11. If $\frac{m}{n} = a$ then $x^{\frac{m}{n}} = x^a$.

Example 5.3.12.

- $\left(\frac{25}{16}\right)^{\frac{-1}{2}} = \left(\sqrt[2]{\frac{25}{16}}\right)^{-1} = \left(\frac{\sqrt{25}}{\sqrt{16}}\right)^{-1} = \left(\frac{5}{4}\right)^{-1} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$
- $\sqrt[4]{7^8} = 7^{\frac{8}{4}} = 7^2 = 49$
- $\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = \left(\left(\sqrt[3]{x}\right)^4\right)^{\frac{3}{4}} = \left(\sqrt[4]{\left(\sqrt[3]{x}\right)^4}\right)^3 = \left|\sqrt[3]{x}\right|^3 = \left|\left(\sqrt[3]{x}\right)^3\right| = |x|$
- $\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} \neq x^{\frac{4}{3} \cdot \frac{3}{4}} = x^{\frac{12}{12}} = x$

Exercise. Evaluate and simplify the following expressions.

a) $4^{\frac{3}{2}}$

b) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$

c) $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

d) $\left(\frac{8}{27}\right)^{\frac{1}{3}}$

e) $\sqrt[16]{3^{32}}$

f) $\sqrt[7]{2^{21}}$

Example 5.3.13.

- $x^{\frac{3}{2}} = 64 \implies \left(\sqrt[2]{x}\right)^3 = 64$

$$\implies \sqrt[3]{\left(\sqrt[2]{x}\right)^3} = \sqrt[3]{64}$$
$$\implies \sqrt[2]{x} = 4$$
$$\implies \left(\sqrt[2]{x}\right)^2 = 16$$
$$\implies x = 16$$

- $x^{\frac{2}{3}} = 4 \implies \left(\sqrt[3]{x}\right)^2 = 4$

$$\implies \sqrt[2]{\left(\sqrt[3]{x}\right)^2} = 2$$
$$\implies \left|\sqrt[3]{x}\right| = 2$$
$$\implies \left|\sqrt[3]{x}\right|^3 = 8$$
$$\implies \left|\left(\sqrt[3]{x}\right)^3\right| = 8$$
$$\implies |x| = 8$$
$$\implies x = \pm 8$$

Exercise. Solve the following equations.

a) $x^{\frac{3}{4}} = 8$

b) $x^{\frac{4}{3}} = 16$

c) $x^{\frac{1}{2}} = 25$

6.1 Introduction to Exponential and Logarithmic Functions

Definition 6.1.1. Let $b \in (0,1) \cup (1,\infty)$. We call functions of the form $f(x) = b^x$ *exponential function*, and b is called the *base* of the function.

Example 6.1.2.

- $f(x) = 2^x$ is a (base 2) exponential function.
- $g(x) = (\frac{1}{2})^x$ is a (base $\frac{1}{2}$) exponential function
- $h(x) = 1^x$ is not an exponential function because $1 \notin (0,1) \cup (1,\infty)$.
- $p(x) = (-3)^x$ is not an exponential function because $-3 \notin (0,1) \cup (1,\infty)$.

Fact 6.1.3 (Properties of graphs of exponential functions)**.** Let $f(x) = b^x$ be an exponential function.

(1) $\text{dom}(f) = (-\infty,\infty)$ and $\text{ran}(f) = (0,\infty)$.

(2) $f(x) > 0$ on $(-\infty,\infty)$.

(3) There are no x-intercepts.

(4) The y-intercept is $(0,1)$.

(5) If $b \in (1,\infty)$ then

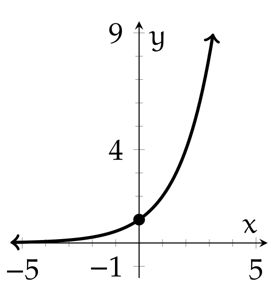
- $f(x)$ is increasing on $(-\infty,\infty)$,
- as $x \to -\infty$, $f(x) \to 0$, and as $x \to \infty$, $f(x) \to \infty$.

(6) If $b \in (0,1)$ then

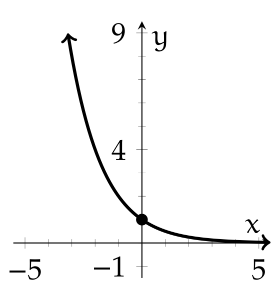
- $f(x)$ is decreasing on $(-\infty,\infty)$,
- as $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to 0$.

Example 6.1.4.

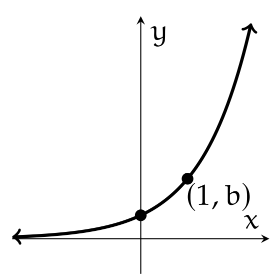
$y = 2^x$



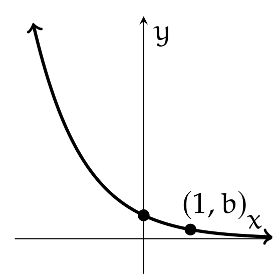
$y = \frac{1}{2}^x$



$y = b^x$ if $b \in (1,\infty)$



$y = b^x$ if $b \in (0,1)$



Remark 6.1.5. The graphs of exponential functions pass the horizontal line test, and hence are invertible. The inverses are what we call exponential functions.

Definition 6.1.6. Let $f(x) = b^x$ be an exponential function. We write $\log_b(x) = f^{-1}(x)$ for the inverse of $f(x)$. We call $\log_b(x)$ the *logarithmic function* of base b.

Note 6.1.7. Recall, if $f^{-1}(x)$ is the inverse of $f(x)$. Then

- $f(x) = y \iff f^{-1}(y) = x$
- $f^{-1}(f(x)) = x$ for all $x \in \text{dom}(f)$
- $f(f^{-1}(x)) = x$ for all $x \in \text{dom}(f^{-1})$

Therefore,

- $b^x = y \iff \log_b(y) = x$
- $\log_b(b^x)) = x$ for all $x \in (-\infty, \infty)$
- $b^{\log_b(x)} = x$ for all $x \in \text{dom}(\log_b)$

Example 6.1.8. Evaluating logarithmic expressions is straightforward using these inverse properties.

- $\log_3(9) = \log_3(3^2) = 2$
- $\log_4(4^{142}) = 142$
- $2^{\log_2(8)} = 2^{\log_2(2^3)} = 2^3 = 8$
- $\log_2\left(\frac{1}{8}\right) = \log_2\left(\frac{1}{2^2}\right) = \log_2\left(2^{-2}\right) = -2$
- $\log_7\left(\sqrt{7}\right) = \log_7\left(7^{1/2}\right) = \frac{1}{2}$
- $5^{\log_5(11)} = 11$

Note 6.1.9. Recall, $\text{dom}(f^{-1}) = \text{ran}(f)$ and $\text{ran}(f^{-1}(x)) = \text{dom}(f)$. Thus, since $\log_b$ is the inverse of the exponential function $f(x) = b^x$, we have

$$\text{dom}(\log_b) = \text{ran}(f) = (0, 1) \quad \text{and} \quad \text{ran}(\log b) = \text{dom}(f) = (-\infty, \infty).$$

Fact 6.1.10 (Properties of graphs of logarithmic functions)**.** Let $\log_b$ be a logarithmic function.

(1) $\text{dom}(\log_b) = (0, \infty)$ and $\text{ran}(\log_b) = (-\infty, \infty)$.

(2) There is no y-intercept, and $(1, 0)$ is the only x-intercept

(3) If $b \in (1, \infty)$ then

- $f(x)$ is increasing on $(0, \infty)$ (everywhere where it defined),
- as $x \to \infty$, $f(x) \to \infty$, and as $x \to 0^+$, $f(x) \to -\infty$

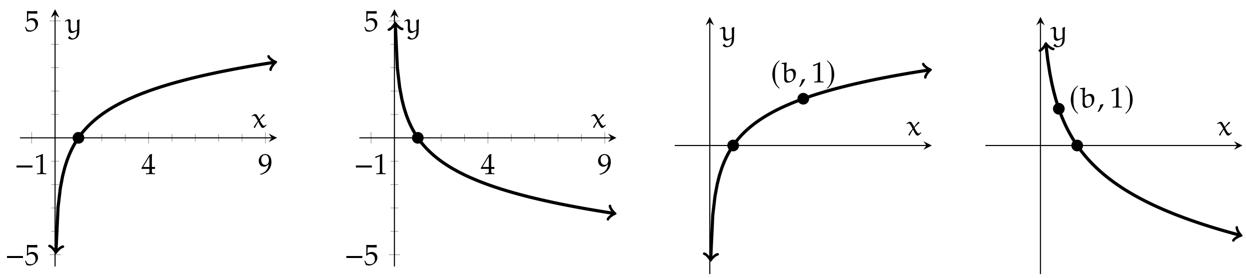
(4) If $b \in (0, 1)$ then

- $f(x)$ is decreasing on $(0, \infty)$ (everywhere where it defined),

- as $x \to \infty$, $f(x) \to -\infty$, and as $x \to 0^+$, $f(x) \to \infty$

Example 6.1.11.

$y = \log_2(x)$ $\quad$ $y = \frac{1}{2}^x$ $\quad$ $y = b^x$ if $b \in (1, \infty)$ $\quad$ $y = b^x$ if $b \in (0, 1)$



Remark 6.1.12.

(1) The function $\log_{10}(x)$ is called the <u>*common log*</u> and is usually written $\log(x)$.

(2) The function $\log_e(x)$ is called the <u>*natural log*</u> and is usually written $\ln(x)$.

Example 6.1.13.

- $\log(100) = \log_{10}(100) = \log_{10}(10^2) = 2$
- $\ln(e^{17}) = \log_e(e^{17}) = 17$
- $\log(0.1) = \log_{10}\left(\frac{1}{10}\right) = \log_{10}(10^{-1}) = -1$
- $e^{\ln(4x-1)} = e^{\log_e(4x-1)} = 4x - 1$

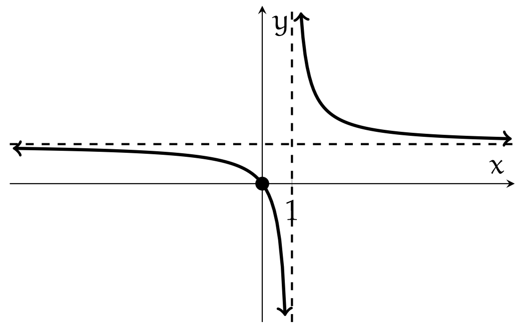
Remark 6.1.14. Recall, the domain of the square root function $\text{sqrt}(x) = \sqrt{x}$ is $[0, \infty)$. Since logarithmic functions have a similar domain, i.e., $(0, \infty)$, finding domains involving logarithms is similar to finding domains involving square roots.

Example 6.1.15.

- Let $f(x) = 3 + \sqrt{x - 4}$. Observe $x - 4 \geqslant 0 \iff x \geqslant 4 \implies \text{dom}(f) = [4, \infty)$.
- Let $g(x) = 3 + \log_5(x - 4)$. Observe $x - 4 > 0 \iff x > 4 \implies \text{dom}(g) = (4, \infty)$.
- Let $h(x) = 2\log(3 - x) - 1$. Observe, $3 - x > 0 \iff 3 > x \implies \text{dom}(h) = (-\infty, 3)$.

Remark 6.1.16. In general, if your function looks like $f(x) = \log(g(x))$ for some $g(x)$, then you find $\text{dom}(f)$ by finding where $g(x) > 0$.

Example 6.1.17. Let $f(x) = \ln\left(\frac{x}{x-1}\right)$. It is hard to solve $\frac{x}{x-1} > 0$ directly, but we can sketch the graph of $g(x) = \frac{x}{x-1}$ to find where $g(x) > 0$. Using our tools to sketch rational functions, observe tht the graph $y = g(x)$ has an x-intercept at $(0, 0)$, a vertical asymptote at $x = 1$, and the same end behavior as $y = 1$. We get the following sketch:



We see that $g(x) > 0$ on $(-\infty, 0) \cup (1, \infty)$. Hence, the domain of g is $(-\infty, 0) \cup (1, \infty)$.

Example 6.1.18. Finding inverses of logarithmic functions and exponential functions is straightforward using the fact that they are each others inverses.

- Let $f(x) = 2^{x-1} - 3$. Then

$$\begin{aligned} y = 2^{x-1} - 3 &\implies x = 2^{y-1} - 3 \\ &\implies x + 3 = 2^{y-1} && \text{(add 3 to both sides)} \\ &\implies \log_2(x+3) = \log_2(2^{y-1}) && \text{(take } \log_2 \text{ on both sides)} \\ &\implies \log_2(x+3) = y - 1 && (\log_2 \text{ is the inverse of } 2^x) \\ &\implies \log_2(x+3) + 1 = y && \text{(add 1 to both sides)} \end{aligned}$$

  Hence, $f^{-1}(x) = \log_2(x + 3) + 1$

- Let $g(x) = \log_7(4x + 8)$. Then

$$\begin{aligned} y = \log_7(4x+8) &\implies x = \log_7(4y+8) \\ &\implies 7^x = 7^{\log_7(4y+8)} && \text{(take } 7^x \text{ on both sides)} \\ &\implies 7^x = 4y + 8 && (7^x \text{ is the inverse of } \log_7) \\ &\implies 7^x - 8 = 4y && \text{(subtract 8 from both sides)} \\ &\implies \frac{7^x - 8}{4} = y && \text{(divide by 4 on both sides)} \end{aligned}$$

  Hence, $g^{-1}(x) = \dfrac{7^x - 8}{4}$.

6.2 Properties of logarithms

Fact 6.2.1. Let $b \in (0,1) \cup (1,\infty)$ and $x, y \in (-\infty,\infty)$.

$$b^{x+y} = b^x b^y \qquad \textbf{(Product Rule)}$$

$$b^{x-y} = \frac{b^x}{b^y} \qquad \textbf{(Quotient Rule)}$$

$$b^{xy} = (b^x)^y \qquad \textbf{(Power Rule)}$$

Remark 6.2.2. Recall the following inverse properties of logarithms and exponential functions:

- $b^x = y \iff \log_b(y) = x$
- $\log_b(b^x) = x$ for all $x \in \text{dom}(b^x) = (-\infty,\infty)$
- $b^{\log_b(x)} = x$ for all $x \in \text{dom}(\log_b) = (0,\infty)$

where $b \in (0,1) \cup (1,\infty)$. We can use these to derive rules for logarithms, for example:

$$\log_b(xy) = \log_b\left(b^{\log_b(x)} b^{\log_b(y)}\right) = \log_b\left(b^{\log_b(x)+\log_b(y)}\right) = \log_b(x) + \log_b(y),$$

so $\log_b(xy) = \log_b(x) + \log_b(y)$.

Fact 6.2.3. Let $b \in (0,1) \cup (1,\infty)$ and $x, y \in (0,\infty)$.

$$\log_b(xy) = \log_b(x) + \log_b(y) \qquad \textbf{(Product Rule)}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \qquad \textbf{(Quotient Rule)}$$

$$\log_b(x^y) = \log_b(x) \cdot y \qquad \textbf{(Power Rule)}$$

Example 6.2.4. The previous fact allows us to rewrite logarithmic expressions.

- $\log_3(9a) = \log_3(9) + \log_3(a) = \log_3(3^2) + \log_3(x) = 2 + \log_3(a)$
- $\log_2\left(\frac{8}{x}\right) = \log_2(8) - \log_2(x) = \log_2(2^3) - \log_2(x) = 3 - \log_2(x)$
- $\ln\left(\left(\frac{3x}{e}\right)^2\right) = \ln\left(\frac{3x}{e}\right) \cdot 2 = (\ln(3x) - \ln(e)) \cdot 2 = \ln(3x) \cdot 2 - \ln(e) \cdot 2 = (\ln(3) + \ln(x)) \cdot 2 - \ln(e) \cdot 2$
  $$= \ln(3) \cdot 2 + \ln(x) \cdot 2 - 1 \cdot 2$$
  $$= \ln(3) \cdot 2 + \ln(x) \cdot 2 - 2$$
- $\log\left(\frac{\sqrt{x}}{yz^3}\right) = \log\left(\sqrt{x}\right) - \log\left(yz^3\right) = \log\left(x^{\frac{1}{2}}\right) - \left(\log(y) + \log\left(z^3\right)\right) = \frac{1}{2}\log(x) - \log(y) - 3\log(z)$

Exercise. Expand and simplify the following using the logarithm properties.

a) $\log_2(4x)$ b) $\log_3(x^4y^2)$ c) $\log_4(x^2-2x-8)$ (Hint: factor)

Example 6.2.5. Use the properties of logarithms to write the following as a single logarithm.

- $\log_2(x) + \log_2(y+1) = \log_2(x(y+1)) = \log_2(xy+y)$
- $\log_3(x-1) - \log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$
- $\log(x) + 2\log(y) - \log(z) = \log(x) + \log\left(y^2\right) - \log(z) = \log\left(xy^2\right) - \log(z) = \log\left(\frac{xy^2}{z}\right)$
- $1 - 3\ln(x) = \ln(e^1) - \ln(x)\cdot 3 = \ln(e) - \ln(x^3) = \ln\left(\frac{e}{x^3}\right)$
- $$\begin{aligned}\frac{\log_4(x)}{3} - \log_4(y) + \log_4(z)\cdot 9 &= \log_4(x)\cdot\frac{1}{3} - \log_4(y) + \log_4(z)\cdot 9\\ &= \log_4\left(x^{\frac{1}{3}}\right) - \log_4(y) + \log_4\left(z^9\right)\\ &= \log_4\left(\sqrt[3]{x}\right) - \log_4(y) + \log_4\left(z^9\right)\\ &= \log_4\left(\frac{\sqrt[3]{x}}{y}\right) + \log_4\left(z^9\right)\\ &= \log_4\left(\frac{\sqrt[3]{x}}{y}\cdot z^9\right) = \log_4\left(\frac{\sqrt[3]{x}\cdot z^9}{y}\right)\end{aligned}$$

Exercise. Use the properties of logarithms to write the following as a single logarithm.

a) $4\log_6(x) + 2\log_6(y)$ b) $\log(x) - \frac{1}{3}\log(y)$ c) $3 + \log_2(x)$

Exercise. Use the properties of logarithms to write the following as a single logarithm.

a) $\ln(x^3y^2)$ b) $\log_2\left(\frac{128}{x^2+4}\right)$ c) $\log\left(\frac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$

Exercise (Bonus)**.** Derive the rule $\log_b(x^y) = \log_b(x)\cdot y$ using fact and remark at the start of this section.

6.3 Exponential Equations

Remark 6.3.1. We know how to solve all kinds of equations.

Linear equations.

Simply solve them directly:

$$\begin{aligned} x+2 &= 3x-6 \\ -2x+2 &= -6 \\ -2x &= -8 \\ \frac{1}{-2}\cdot -2x &= \frac{1}{-2}\cdot -8 \\ x &= 4 \end{aligned}$$

Quadratic equations.

Set equal to zero, factor/complete the square:

$$\begin{aligned} 3x+1 &= -2x^2-x+7 \\ 2x^2+3x+1 &= -x+7 \\ 2x^2+4x+1 &= 7 \\ 2x^2+4x-6 &= 0 \\ x^2+2x-3 &= 0 \\ (x+3)(x-1) &= 0 \\ x &= -3,1 \end{aligned}$$

Absolute value equations.

Solve using cases:

$$\begin{aligned} 1-|x+1| &= -4 \\ -|x+1| &= -5 \\ |x+1| &= 5 \\ \pm(x+1) &= 5 \\ x &= -6,4 \end{aligned}$$

Rational equations.

Get rid of denominators, solve as others:

$$\begin{aligned} (x-2) &= \frac{5x-8}{x+2} \\ (x-2)(x+2) &= 5x-8 \\ x^2-4 &= 5x-8 \\ x^2-5x+4 &= 0 \\ (x-1)(x-4) &= 0 \\ x &= 1,4 \end{aligned}$$

Radical equations.

Translate into radicals, use radical/exponent identities to get rid of them, then solve as others:

$$\begin{aligned} x^{2/3} &= 4 \\ \left(\sqrt[3]{x}\right)^2 &= 4 \\ \sqrt[2]{\left(\sqrt[3]{x}\right)^2} &= \sqrt[2]{4} \\ \left|\sqrt[3]{x}\right| &= 2 \\ \sqrt[3]{x} &= \pm 2 \\ \left(\sqrt[3]{x}\right)^3 &= (\pm 2)^3 \\ x &= \pm 8 \end{aligned}$$

Example 6.3.2. To solve exponential equations, we apply their inverses (logarithms):

- $$\begin{aligned} 3^{x-4} &= 8 \\ \log_3(3^{x-4}) &= \log_3(8) \\ x-4 &= \log_3(8) \\ x &= \log_3(8)+4 \end{aligned}$$

- $$\begin{aligned} 5^x - 11 &= 3 \\ 5^x &= 14 \\ \log_5(5^x) &= \log_5(14) \\ x &= \log_5(14) \end{aligned}$$

- $$\begin{aligned} 2^{x-1} &= 2^{5x+3} \\ \log_2\left(2^{x-1}\right) &= \log_2\left(2^{5x+3}\right) \\ x-1 &= 5x+3 \\ -4x-1 &= 3 \\ -4x &= 4 \\ x &= -1 \end{aligned}$$

- $$\begin{aligned} 3^{2x} &= 9^{1-x} \\ 3^{2x} &= \left(3^2\right)^{1-x} \\ 3^{2x} &= 3^{2\cdot(1-x)} \\ 3^{2x} &= 3^{2-2x} \\ \log_3\left(3^{2x}\right) &= \log_3\left(3^{2-2x}\right) \\ 2x &= 2-2x \\ 4x &= 2 \\ x &= 2 \end{aligned}$$

- $$\begin{aligned} 4 &= 2\cdot 3^{4x-1} \\ 2 &= 3^{4x} \\ \log_3(2) &= \log_3\left(3^{4x-1}\right) \\ \log_3(2) &= 4x-1 \\ \log_3(2)+1 &= 4x \\ \frac{\log_3(2)+1}{4} &= x \end{aligned}$$

- $$\begin{aligned} 6^{7x-1} &= 3^x \\ \log_6\left(6^{7x-1}\right) &= \log_6\left(3^x\right) \\ 7x-1 &= \log_6\left(3^x\right) \\ 7x-1 &= \log_6(3)x \\ 7x-1-\log_6(3)x &= 0 \\ 7x-\log_6(3)x &= 1 \\ (7-\log_6(3))x &= 1 \\ x &= \frac{1}{7-\log_6(3)} \end{aligned}$$

- $$\begin{aligned} 5^{x-1} &= 13^{x+1} \\ \log_5\left(5^{x-1}\right) &= \log_5\left(13^{x+1}\right) \\ x-1 &= \log_5(13)\cdot(x+1) \\ x-1 &= \log_5(13)\cdot x+\log_5(13)\cdot 1 \\ x-1 &= \log_5(13)\cdot x+\log_5(13) \\ x-\log_5(13)x &= \log_5(13)+1 \\ (1-\log_5(13))x &= \log_5(13)+1 \\ x &= \frac{\log_5(13)+1}{1-\log_5(13)} \end{aligned}$$

- $$\begin{aligned} \frac{3^x}{7^{x+2}} &= 1 \\ 3^x &= 7^{x+2} \\ \log(3^x) &= \log(7^{x+2}) \\ \log(3)x &= \log(7)(x+2) \\ \log(3)x &= \log(7)x+\log(7)2 \\ \log(3)x-\log(7)x &= \log(7)2 \\ (\log(3)-\log(7))x &= \log(7)2 \\ x &= \frac{\log(3)-\log(7)}{\log(7)2} \end{aligned}$$

- $$\begin{aligned} e^{2x} - 2e^x &= 0 \\ e^{2x} &= 2e^x \\ \ln\left(e^{2x}\right) &= \ln(2e^x) \\ 2x &= \ln(2) + \ln(e^x) \\ 2x &= \ln(2) + x \\ x &= \ln(2) \end{aligned}$$

- $$\begin{aligned} e^{-5x} &= \frac{1}{2} \\ \ln\left(e^{-5x}\right) &= \ln\left(\frac{1}{2}\right) \\ -5x &= \ln\left(\frac{1}{2}\right) \\ x &= \frac{\ln\left(\frac{1}{2}\right)}{-5} \\ x &= \frac{\ln(1) - \ln(2)}{-5} \\ x &= \frac{\ln(e^0) - \ln(2)}{-5} \\ x &= \frac{-\ln(2)}{-5} \\ x &= \frac{\ln(2)}{5} \end{aligned}$$

Exercise. Solve the following equations.

a) $3^{x-1} = 9$ b) $2^{7x} = 8^{4-2x}$ c) $10^{x-1} = 4^{3x-1}$ d) $5^{x-3} - 7^{1-x} = 0$

6.4 Logarithmic Equations

Example 6.4.1.

- $$\begin{aligned} \log_5(1-3x) &= \log_5(x^2-3) \\ 5^{\log_5(1-3x)} &= 5^{\log_5(x^2-3)} \\ 1-3x &= x^2-3 \\ -3x &= x^2-4 \\ 0 &= x^2+3x-4 \\ x^2+3x-4 &= 0 \\ (x+4)(x-1) &= 0 \\ x &= -4, 1 \\ x &= -4 \end{aligned}$$

- $$\begin{aligned} \log_6(x+4)+\log_6(3-x) &= 1 \\ \log_6((x+4)(3-x)) &= 1 \\ 6^{\log_6((x+4)(3-x))} &= 6^1 \\ (x+4)(3-x) &= 6 \\ -x^2+12-x &= 6 \\ -x^2-x+6 &= 0 \\ x^2+x-6 &= 0 \\ (x+3)(x-2) &= 0 \\ x &= -3, 2 \end{aligned}$$

- $$\begin{aligned} 2-\ln(x-3) &= 1 \\ -\ln(x-3) &= -1 \\ \ln(x-3) &= 1 \\ e^{\ln(x-3)} &= e^1 \\ x-3 &= e \\ x &= e+3 \end{aligned}$$

- $$\begin{aligned} \log_{\frac{1}{7}}(x) &= -2 \\ \left(\frac{1}{7}\right)^{\log_{\frac{1}{7}}(x)} &= \left(\frac{1}{7}\right)^{-2} \\ x &= \frac{1}{\left(\frac{1}{7}\right)^2} \\ x &= \frac{1}{\frac{1}{49}} \\ x &= 49 \end{aligned}$$

- $$\begin{aligned} \log_{64}\left(\frac{9x-2}{2x+3}\right) &= \frac{1}{3} \\ 64^{\log_{64}\left(\frac{9x-2}{2x+3}\right)} &= 64^{\frac{1}{3}} \\ \frac{9x-2}{2x+3} &= \sqrt[3]{64} \\ \frac{9x-2}{2x+3} &= 4 \\ 9x-2 &= 4(2x+3) \\ 9x-2 &= 8x+12 \\ x &= 14 \end{aligned}$$

- $$\begin{aligned} \log(x)-\log(2) &= \log(x+8)-\log(x+2) \\ \log\left(\frac{x}{2}\right) &= \log\left(\frac{x+8}{x+2}\right) \\ 10^{\log\left(\frac{x}{2}\right)} &= 10^{\log\left(\frac{x+8}{x+2}\right)} \\ \frac{x}{2} &= \frac{x+8}{x+2} \\ x^2+2x &= 2x+16 \\ x^2-16 &= 0 \\ (x-4)(x+4) &= 0 \\ x &= -4, 4 \\ x &= 4 \end{aligned}$$

Exercise. Solve the following equations for x.

a) $\log_5(3x-4) = \log_5(6-2x)$

b) $\log_2(x^2+3x) = 2$

c) $\ln(x^2-24) = 0$

d) $\log_4(x-3)+\log_4(x+3) = 2$

6.5 Applications of Exponential/Logarithmic Functions

Example 6.5.1. Suppose a patient in a hospital has a tumor. The tumor's diameter in millimeters grows according to the formula $D(t) = 10 \cdot 4^{t/24}$ where t is the number of hours after it is detected.

- What was the diameter of the tumor 24 hours after it was detected?
  $D(24) = 10 \cdot 4^{24/24} = 10 \cdot 4^1 = 40$ mm

- What was the diameter of the tumor when it was detected?
  $D(0) = 10 \cdot 4^{0/24} = 10 \cdot 4^0 = 10 \cdot 1 = 10$ mm

- When will the diameter be 160 mm?

$$\begin{aligned} D(t) &= 160 \\ 10 \cdot 4^{t/24} &= 160 \\ 4^{t/24} &= 16 \\ \log_4\left(4^{t/24}\right) &= \log_4(16) \\ t/24 &= 2 \\ t &= 48 \text{ hours} \end{aligned}$$

- How long does it take for the diameter of the tumor to double?

$$\begin{aligned} D(t) &= 20 \\ 10 \cdot 4^{t/24} &= 20 \\ 4^{t/24} &= 2 \\ \log_4\left(4^{t/24}\right) &= \log_4(2) \\ \log_4\left(4^{t/24}\right) &= \log_4(4^{1/2}) \\ t/24 &= 1/2 \\ t &= 12 \text{ hours} \end{aligned}$$

- How long does it take for the diameter of the tumor to triple?

$$\begin{aligned} D(t) &= 30 \\ 10 \cdot 4^{t/24} &= 30 \\ 4^{t/24} &= 3 \\ \log_4\left(4^{t/24}\right) &= \log_4(3) \\ t/24 &= \log_4(3) \\ t &= \log_4(3)/24 \text{ hours} \end{aligned}$$

Example 6.5.2. The population of undercover aliens in Las Cruces is modeled by

$$P(t) = \frac{50}{1 + 4^{-\frac{1}{3}t}}$$

where t is the number of years after 2020.

- Find and interpret the meaning of $P(0)$.

$$P(0) = \frac{50}{1 + 4^{-\frac{1}{3}\cdot 0}} = \frac{50}{1 + 4^0} = \frac{50}{2} = 25$$

  There are 25 undercover aliens in Las Cruces in 2020.

- When will the population of undercover aliens in Las Cruces reach 40?

$$\begin{aligned} P(t) &= 40 \\ \frac{50}{1 + 4^{-\frac{1}{3}t}} &= 40 \\ 50 &= 40(1 + 4^{-\frac{1}{3}t}) \\ 50 &= 40 + 40 \cdot 4^{-\frac{1}{3}t} \\ 10 &= 40 \cdot 4^{-\frac{1}{3}t} \\ \frac{1}{4} &= 4^{-\frac{1}{3}t} \\ \log_4\left(\frac{1}{4}\right) &= \log_4\left(4^{-\frac{1}{3}t}\right) \\ -1 &= -\frac{1}{3}t \\ 3 &= t, \text{ so in 2023} \end{aligned}$$

Example 6.5.3. Ethanol is a drug used to increase happiness and sociability. It decays in the bloodstream according to the formula $E(t) = 2e^{-0.02t}$, where $E(0) = 4$ is the initial amount of the drug in the bloodstream. Find an expression only for the time until half of the Ethanol remains in the bloodstream.

Need to find t when $E(t) = 2$:

$$\begin{aligned} 4e^{-0.003t} &= 2 \\ e^{-0.02t} &= \frac{1}{2} \\ \ln\left(e^{-0.02t}\right) &= \ln\left(2^{-1}\right) \\ -0.02t &= \ln(2) \cdot -1 \\ 0.02t &= \ln(2) \\ t &= 50\ln(2) \end{aligned}$$

0.1 Sets and their notation

Definition 0.1.1. A _set_ is a well-defined collection of objects which are called the _elements_ of the set.

Remark 0.1.2. There are multiple ways to describe a specific set.

(1) The collection of letters in the word "coffee" – carefully describing the set in words.

(2) $\{c, o, f, e\}$ – the roster method; explictly listing the elements between curly brackets.

(3) $\{\ell \mid \ell$ is a letter in the word "coffee"$\}$ – the set builder method; describe the set by the condition elements x need to satisfy to be in the set.

Note that (1)-(3) all describe the same set.

Remark 0.1.3. A set must either contain an element or not. Otherwise, it is not a well-defined collection. We use the symbol $\in$ (and $\notin$) to describe that an element is (or is not) in a set. For example, let $C = \{c, o, f, e\}$ be the set from the previous example. Then:

- $c \in C$ since "c" is a letter in the word "coffee."
- $o \in C$ since "o" is a letter in the word "coffee".
- $\ell \notin C$ since "ℓ" is not a letter in the word "coffee."
- $0 \notin C$ since 0 is not a letter in the word "coffee".
- $ff \notin C$ since "ff" is not a letter in the word "coffee."

Example 0.1.4. The following sets will show up relatively often in this class:

- The _empty set_ $\varnothing := \{\}$. Note that $\varnothing$ is not the same as the number 0. (Actually it is)
- The _natural numbers_ $\mathbb{N} := \{1, 2, 3, \dots\}$.
- The _integers_ $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- The _real numbers_ $\mathbb{R} := \{x \mid x \text{ has a decimal representation}\}$.

Remark 0.1.5.

(1) Sets do not "count" elements. For example, there is no difference between the sets $A = \{a\}$ and $B = \{a, a\}$. Both sets contain only the element a, that is, $a \in A$ and $a \in B$, and $\ell \notin A$ and $\ell \notin B$ for any $\ell \neq a$. Therefore, it is uncommon to repeat an element when writing a set.

(2) Sets are "unordered." For example, there is no difference between the sets $\{1, 2\}$ and $\{2, 1\}$.

Definition 0.1.6. Suppose A and B are sets.

- The *intersection* of A and B is the set of elements containing all the elements that are both in A and B. Formally, $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$.
- The *union* of A and B is the set of elements containing all the elements that are either in A or B (or both). Formally, $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$.

Example 0.1.7. Let $A = \{c, o, f, e\}$ be the collection of letters in the word "coffee" and $B = \{e, f, c, t\}$ the collection of letters in the word "effect". Then

$$A \cap B = \{c,f,e\} \qquad and \qquad A \cup B = \{c,o,f,e,t\}. \tag{1}$$

Exercise. Expand the following unions/intersections.

a) $\{0, 1, 4\} \cap \{-1, 4, 5, 6\}$ b) $\{-2\} \cup \{4, -2\}$ c) $\mathbb{N} \cap \{-4, -2, 1, 0\}$