How to derive well-known dualities in point-free topology from Priestley duality

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There are several prominent duality results in pointfree topology.

- The Hofmann-Lawson duality (1978) between the category of **continuous frames** and the category of **locally compact sober spaces**.
- The duality for stably continuous frames between the category of **stably continuous frames** and the category of **stably locally compact spaces** ([Gierz-Keimel, 1977], [Banaschewski, 1981], [Johnstone, 1981], [Simmons, 1982]).
- The Isbell duality (1972) between the category of **compact regular frames** and the category of **compact Hausdorff spaces**.

We will give a new proof of these results using Priestley duality.

Priestley duality (for frames)

Priestley duality

A Priestley space is a compact space X with a partial ordering $\leq$ satisfying the Priestley separation axiom:

$$x \not\leq y \implies \exists \text{clopen upset } U : x \in U \text{ and } y \notin U.$$

Consider the following categories.

- **Pries** Priestley spaces and continuous order-preserving maps.
- **DLat** bounded distributive lattices and bounded lattice homomorphisms.

Theorem (Priestley, 1970)

Pries is dually equivalent to **DLat**

DLat $\longleftrightarrow$ Pries

L-spaces

Frames are bounded distributive lattices that satisfy $a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$.

Hence, they correspond to special Priestley spaces.

- An **Esakia space** is a Priestley space with the property that the downsets of clopen sets are clopen.
- An Esakia space is **extremally order-disconnected** if the closure of every open upset is open.
- An **L-space** (localic space) is an extremally order-disconnected Esakia space.

Lemma

$D \in \mathbf{DLat}$ is a frame iff its Priestley dual X_D is an L-space

L-morphisms

Similarly, frame homomorphisms are special bounded lattice homomorphisms, so they correspond to special order-preserving continuous maps.

An **L-morphism** is a continuous order-preserving map $f : X_1 \to X_2$ between Priestley spaces such that $\text{cl}(f^{-1}(U)) = f^{-1}(\text{cl}(U))$ for all open upsets U of X_2.

If f is the dual of a bounded lattice homomorphism h then we have:

Lemma

h is a frame homomorphism iff f is an L-morphism

Pultr-Sichler duality

Consider the following categories.

Frm frames and frame homomorphisms.

LPries L-spaces and L-morphisms.

Theorem (Pultr-Sichler, 1988)

The functors establishing Priestley duality restrict to **Frm** and **LPries**

$$
\begin{array}{ccc}
\mathsf{DLat} & \leftrightsquigarrow & \mathsf{Pries} \\
\curlyvee & & \curlyvee \\
\mathsf{Frm} & \leftrightsquigarrow & \mathsf{LPries}
\end{array}
$$

Spatiality

A frame L is spatial if it is isomorphic to the frame of opens of a space.

In the language of the Priestley duality we have the following.

Theorem

L is spatial iff $Y_L = \{x \in X_L \mid {\downarrow}x \text{ is } \textit{clopen}\}$ is dense in X_L

SL-spaces

An SL-space (spatial L-space) is an L-space X such that $Y := \{x \in X \mid {\downarrow}x \text{ is clopen}\}$ is dense in X.

Consider the following categories.

- **SL** the full subcategory of **LPries** containing SL-spaces.
- **SFrm** the full subcategory of **Frm** containing spatial frames.

Theorem

The functors establishing Priestley duality restrict to **SL** and **SFrm**

Sober spaces

Let Sob be the category of sober spaces.

Theorem

There is an equivalence between **SL** and **Sob**

Consequently, we obtain the fact that **Sob** is dual to **SFrm**.

It is the map $X \mapsto Y = \{x \in X \mid {\downarrow}x \text{ is clopen}\}$ that establishes this theorem. Here, we view Y as a space where the opens are exactly the clopen upsets of X intersected down to Y.

In fact, the space of points of a frame L is exactly the space Y_L.

Theorem

Y_L is homeomorphic to the points of L

$$\begin{array}{ccccc} \mathsf{DLat} & \longleftrightsquigarrow & \mathsf{Pries} & & \\ \curlyvee & & \curlyvee & & \\ \mathsf{Frm} & \longleftrightsquigarrow & \mathsf{LPries} & & \\ \not\vee & & \not\vee & & \\ \mathsf{SFrm} & \longleftrightsquigarrow & \mathsf{SL} & \longleftrightarrow & \mathsf{Sob} \end{array}$$

Deriving Hofmann-Lawson duality

Hofmann-Lawson duality

A frame L is **continuous** if $a = \bigvee\{b \in L : b \ll a\}$ for each $a \in A$, where $\ll$ is the **way-below** relation. A frame homomorphism is **proper** if it preserves $\ll$.

A continous map $f : X \to Y$ is **proper** it $\downarrow f(A)$ is closed for every closed $A \subseteq X$, and $f^{-1}(B)$ is compact for every compact saturated set $B \subseteq Y$.

CFrm continuous frames and proper frame homomorphisms.

LCSob locally compact sober spaces and proper continuous maps.

Theorem (Hofmann-Lawson, 1978)

CFrm is dually equivalent to **LCSob**

Kernels of clopen upsets

Let X be an L-space, and U, V clopen upsets of X. We write $U \ll V$ if

$$\forall W \in \mathrm{OpUp}(X) : V \subseteq \mathrm{cl}(W) \implies U \subseteq W$$

The **kernel** of U is $\ker(U) = \bigcup\{V \in \mathrm{ClopUp}(X) \mid V \ll U\}$.

Lemma

Suppose X is the Priestley space of the frame L. Then

$$a \ll b \text{ iff } \varphi(a) \ll \varphi(b) \text{ iff } \ker\varphi(a) \subseteq \varphi(b)$$

for all $a, b \in L$

Here $\varphi : L \to \mathrm{ClopUp}(X_L)$ is the **Stone map** defined by $\varphi(a) = \{x \in X_L \mid a \in x\}$.

Packed L-spaces

Let X be an L-space.

- $U \in \mathrm{ClopUp}(X)$ is **packed** if $\ker(U)$ is dense in U.
- X is **heriditarily packed** if every $U \in \mathrm{ClopUp}(X)$ is packed.
- An L-morphism $f : X \to Z$ is **proper** iff

$$f^{-1}(\ker U) \subseteq \ker f^{-1}(U)$$

for every clopen upset $U \subseteq Z$.

Let **HPL** be the category of heriditarily packed L-spaces and proper L-morphisms.

Deriving Hofmann-Lawson duality

Theorem

The functors establishing Priestley duality restrict to **CFrm** and **HPL**

Theorem

The functors establishing the duality between **SL** and **Sob** restrict to **HPL** and **LCSob**

Hofmann-Lawson duality is an immediate consequence.

$$
\begin{array}{ccccc}
\mathsf{DLat} & \leftrightsquigarrow & \mathsf{Pries} & & \\
\curlyvee & & \curlyvee & & \\
\mathsf{Frm} & \leftrightsquigarrow & \mathsf{LPries} & & \\
\rotatebox{-90}{$\nleq$} & & \rotatebox{-90}{$\nleq$} & & \\
\mathsf{SFrm} & \leftrightsquigarrow & \mathsf{SL} & \longleftrightarrow & \mathsf{Sob} \\
\curlyvee & & \curlyvee & & \curlyvee \\
\mathsf{CFrm} & \leftrightsquigarrow & \mathsf{HPL} & \longleftrightarrow & \mathsf{LCSob}
\end{array}
$$

Deriving the duality for stably continuous frames

Stably continuous frames

A continuous frame L is **stably continuous** if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$.

A frame L is **compact** if $1 \ll 1$; a frame L is **stably compact** if it is compact and stably continuous.

A space X is **stably locally compact** if it is locally compact, sober, and coherent (binary intersections of compact saturated sets are compact); X is **stably compact** if it is compact and stably locally compact.

The dualities for stably continuous frames

StCFrm full subcategory of **CFrm** containing stably continuous frames.

StKFrm full subcategory of **CFrm** containing stably compact frames.

StLCSp full subcategory of **LCSob** containing stably locally compact spaces.

StKSp full subcategory of **LCSob** containing stably compact spaces.

Theorem (Gierz-Keimel, Banaschewski, Johnstone, Simmons)

1. **StCFrm** is dually equivalent to **StLCSp**
2. **StKFrm** is dually equivalent to **StKSp**

Stability in L-spaces

Let L be frame, X_L its Priestley space, and $a, b \in L$. Then

$$(\forall c \in L)(c \ll a, b \Rightarrow c \ll a \wedge b) \text{ iff } \ker \varphi(a) \cap \ker \varphi(b) = \ker \varphi(a \wedge b)$$

An L-space X is **Scott-stable** iff $\ker U \cap \ker V = \ker(U \cap V)$ for all clopen upsets U, V of X.

Let **SHPL** be the full subcategory of **HPL** containing Scott-stable L-spaces.

Theorem

1. **StCFrm** is dually equivalent to **SHPL**
2. **SHPL** is equivalent to **StLCSp**

Compactness in L-spaces

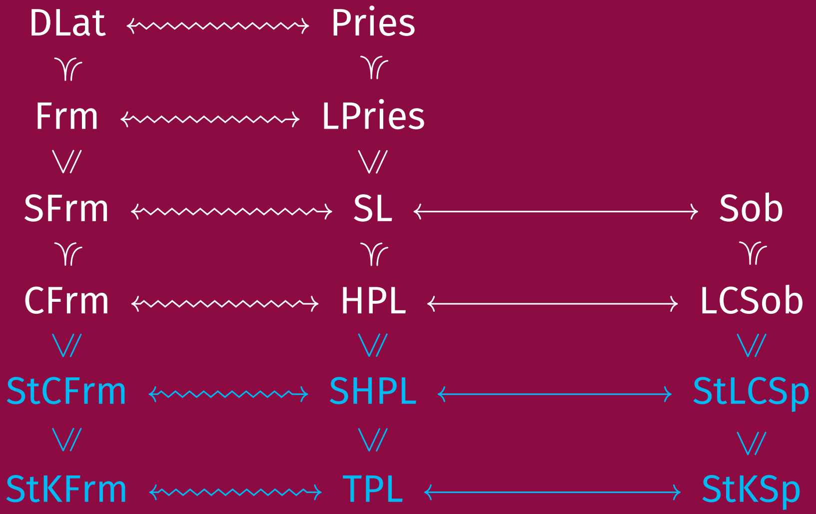
An SHPL-space X is tightly packed iff $\ker X = X$.

Let TPL be the full subcategory of **SHPL** containing tightly packed spaces.

Theorem

1. **StKFrm** is dually equivalent to **TPL**
2. **TPL** is equivalent to **StKSp**

The dualities for stably continuous frames are an immediate consequence of the last two theorems.


DLat
Pries
Frm
LPries
SFrm
SL
Sob
CFrm
HPL
LCSob
StCFrm
SHPL
StLCSp
StKFrm
TPL
StKSp

Deriving Isbell duality

Isbell duality

A frame L is regular if $a = \bigvee\{b \in L \mid b \prec a\}$ for each $a \in L$, where $\prec$ is the well-inside relation, i.e., $a \prec b$ if $a^* \vee b = 1$.

Consider the following categories.

KRFrm full subcategory of **Frm** containing compact regular frames.

KHaus full subcategory of **Sob** containing compact Hausdorff spaces.

Theorem (Isbell, 1972)

KRFrm is dually equivalent to **KHaus**

Regularity in L-spaces

Let X be an L-space and $U \subseteq X$ a clopen upset. Define the **regular part** of U as $\operatorname{reg} U = \bigcup\{V \in \mathsf{ClopUp}(X) \mid {\downarrow}V \subseteq U\}$.

X is **regular** if $\operatorname{reg} U$ is dense in U for every clopen upset U.

Let **RTPL** be the full subcategory of **SL** containing regular tightly packed spaces.

Lemma

Every L-morphism from a tightly packed L-space to a regular L-space is proper

Hence, **RTPL** is a full subcategory of **TPL**. Dually, this implies the well-known fact that every frame homomorphism between compact regular frames is proper.

Deriving Isbell duality

We can restrict our dualities for stably continuous frames to obtain:

Theorem

1. **KRFrm** is dually equivalent to **RTPL**
2. **RTPL** is equivalent to **KHaus**

Isbell duality is an immediate consequence.

$$
\begin{array}{ccccc}
\mathsf{DLat} & \leftrightsquigarrow & \mathsf{Pries} & & \\
\curlyvee & & \curlyvee & & \\
\mathsf{Frm} & \leftrightsquigarrow & \mathsf{LPries} & & \\
\veebar & & \veebar & & \\
\mathsf{SFrm} & \leftrightsquigarrow & \mathsf{SL} & \longleftrightarrow & \mathsf{Sob} \\
\curlyvee & & \curlyvee & & \curlyvee \\
\mathsf{CFrm} & \leftrightsquigarrow & \mathsf{HPL} & \longleftrightarrow & \mathsf{LCSob} \\
\veebar & & \veebar & & \veebar \\
\mathsf{StCFrm} & \leftrightsquigarrow & \mathsf{SHPL} & \longleftrightarrow & \mathsf{StLCSp} \\
\veebar & & \veebar & & \veebar \\
\mathsf{StKFrm} & \leftrightsquigarrow & \mathsf{TPL} & \longleftrightarrow & \mathsf{StKSp} \\
\veebar & & \veebar & & \veebar \\
\mathsf{KRFrm} & \leftrightsquigarrow & \mathsf{RTPL} & \longleftrightarrow & \mathsf{KHaus}
\end{array}
$$

The role of kernels

Going down the chain of dualities, the map

$$\ker : \mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$$

became more refined.

In short, the following happens:

$X \in \mathbf{HPL}$ implies ker is injective.

$X \in \mathbf{SHPL}$ implies ker is a $\wedge$-homomorphism.

$X \in \mathbf{TPL}$ implies ker is a topological interior.

$X \in \mathbf{RTPL}$ implies ker and reg coincide.

Thank you!