Canonical Formulas for IK4

Sebastian Melzer (sdmelzer@skydivizer.com)

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Master of Logic, ILLC, University of Amsterdam

Canonical formulas

Introduced by Zakharyaschev (1980s-90s) for intermediate and modal logics.

Frame-theoretic and dependent on the exact dual structure of Heyting and $K4$-algebras.

Unclear how to apply it to other logics, for example, intuitionistic modal logic.

G. Bezhanishvili and N. Bezhanishvili (2000s-10s) have developed a uniform algebraic approach relying on locally finite reducts (collaborators include Gabelaia, Ghilardi, Iemhoff, Ilin, Jibladze, de Jongh).

Constructing canonical formulas

Practically, we require two procedures:

1. For each formula ϕ find refutation patterns:
   - Collection of algebras $A, \ldots, A_n$ with parameter sets $D_1, \ldots, D_n$.
   - This is where local finiteness comes in handy.
2. An encoding function α of refutation patterns into formulas.
   - Similar to the construction of Jankov formulas.

Note, ϕ has to be semantically equivalent to $\alpha(A_1, D_1) \wedge \cdots \wedge \alpha(A_n, D_n)$.

Applications of canonical formulas

The method provides a lot of structure in the study of intermediate and modal logics:

- Fundamental instances of canonical formulas characterize logics with good properties, e.g. subframe and cofinal subframe logics.
- Preservation results, e.g. Zakharyaschev obtained a proof for the Dummett-Lemmon conjecture that the least modal companion of a Kripke-complete intermediate logic is Kripke-complete.

Goals of this talk

Discuss canonical formulas for intuitionistic modal logic:

- Stable canonical formulas for *IK*4.
- Steady canonical formulas for *PLL*.

Show canonical formulas at work:

- Defining big classes of logics with the finite model property (fmp).
- Lining out preservation results.

Heyting algebras with modal operators

Recall, a Heyting algebra is a bounded lattice A with a binary operator $\to$ such that

$$c \wedge a \leq b \text{ iff } c \leq a \to b \qquad \text{for all } a, b, c \in A$$

A unary operator $\Box$ on a bounded lattice A is modal iff

$$\Box 1 = 1$$

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad \text{for all } a, b \in A$$

An *IK*-algebra is a Heyting algebra with a modal operator.

Besides, it is an *IK*4-algebra iff

$$\Box a \leq \Box\Box a \qquad \text{for all } a \in A.$$

Refutation patterns

Heyting algebras have two established locally finite reducts:

1. The $\to$-free reduct: distributive bounded lattices.
2. The $\vee$-free reduct: bounded implicative semilattices.

Moreover, given a finite subset of some Heyting algebra, a finitely generated algebra of these reducts can be extended back into a Heyting algebra.

The same can be achieved for *IK*-algebras with the $(\to, \Box)$-free reduct.

Refutation patterns

This method, which is also known as filtration, results in subalgebras of the reduct which “semi”-preserve $\Box$:

Definition – Stability

*A bounded lattice homomorphism $h : A \to B$ between IK-algebras is **stable** iff $h\Box a \leq \Box ha$ for all $a \in A$.*

Basically, for each IK-algebra $B \not\models \phi$ we can find a finite stable subalgebra $A \not\models \phi$.

Additionally, we use parameter sets $D_{\to} \subseteq A^2$ and $D_{\Box} \subseteq A$ to describe crucial parts of the valuation that refutes ϕ.

The tuple $(A, D_{\to}, D_{\Box})$ is a refutation pattern for ϕ.

Encoding function

We encode refutation patterns with a generalization of Jankov formulas.

Instead of encoding the full structure of $IK4$-algebras we only encode the bounded lattice structure fully, and the missing operators over the parameter sets.

Ultimately, we obtain the following:

Theorem

*$B \not\models \alpha(A, D_{\rightarrow}, D_{\Box})$ iff there is a homomorphic image C of B and a **stable** **bounded lattice embedding** from A into C which respects the parameters.*

Stable canonical formulas for $IK4$

Recapping:

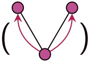
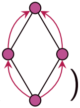
1. We can find refutation patterns using via the local finiteness of $(\Box, \to)$-free IK-algebras.
2. We can encode refutation patterns of $IK4$-algebras into formulas using generalized Jankov formulas.

Theorem

All intuitionistic modal logics extending IK4 are axiomatizable by stable canonical formulas.

Example axiomatizations

$IS4 = IK4 \oplus \alpha(\bullet) \oplus \alpha($  $)$

$IS4.3 = IS4 \oplus \alpha($  $) \oplus \alpha($  $)$

$IK4.C4 = IK4 \oplus \alpha($  $, \varnothing, \{\varnothing\}) \oplus \alpha($  $, \varnothing, \{\{*\}\})$

$PLL = IK4.C4 \oplus \alpha($  $) \oplus \alpha($  $, \varnothing, \{\{*\}\}) \oplus \alpha($  $, \varnothing, \{\uparrow *\})$

N.B.: These are Kripke frames, the “real” axiomatizations use their complex algebras.

Finite model property

Usually, fundamental instances of canonical formulas instantiate classes of logics with the fmp.

Call $\alpha(A) := \alpha(A, \varnothing, \varnothing)$ a stable formula.

Theorem - Stable logics

All transitive intuitionistic modal logics axiomatized by stable formulas have the fmp.

The key is filtration; stable logics are closed under filtration!

Mirroring stability

Recall, Heyting algebras have two established locally finite reducts:

1. The $\to$-free reduct.
2. The $\vee$-free reduct.

In the intermediate setting both generate canonical formulas.

We used the $(\to, \Box)$-free reduct of $IK4$-algebras to obtain stable canonical formulas.

Can we do something similar with the $(\vee, \Box)$-free reduct?

We can but it doesn't work for $IK4$-algebras...

Nuclear algebras

Nuclear algebras are $IK4$-algebras A that satisfy:

$$a \leq \Box a \quad \text{and} \quad \Box\Box a \leq \Box a$$

for each $a \in A$.

They give the algebraic semantics for PLL.

We call any logic that extends PLL a lax logic.

We will use the $(\vee, \Box)$-reduct of nuclear algebras to define steady canonical formulas for lax logics.

Steady canonical formulas

Steady canonical formulas $\beta(A, D_\vee, D_\Box)$ mirror stable canonical formulas in the sense that they encode $\Box$ in the other direction.

Definition – Steadiness

A function $h : A \to B$ between nuclear algebras is ***steady*** *iff $\Box ha \leq h\Box a$ for all $a \in A$.*

Theorem - Refuting steady canonical formulas

$B \not\models \beta(A, D_\vee, D_\Box)$ iff there is a homomorphic image C of B and a ***steady bounded implicative semilattice embedding*** *from A into C that respects the parameters.*

Steady canonical formulas

Theorem

All lax logics are axiomatized by steady canonical formulas.

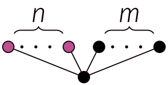
Again, we obtain an fmp result from fundamental instances of these formulas.

A steady formula is a steady canonical formulas of the form $\beta(A) := \beta(A, \varnothing, \varnothing)$.

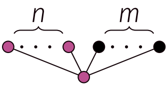
Theorem – Steady logics

All lax logics axiomatized by steady formulas have the fmp.

Examples of steady logics



$\mathfrak{F}_{\bullet}^{n,m}$



$\mathfrak{F}_{\circ}^{n,m}$

$n + m \geq 2$

Theorem

1. *$PLL \oplus \beta(\mathfrak{F}_{\bullet}^{n,m})$ is the lax logic of all finite rooted frames that do not have $n + m$ maximal elements with at least n nuclear.*
2. *$PLL \oplus \beta(\mathfrak{F}_{\circ}^{n,m})$ is the lax logic of all finite $\circ$-rooted frames that do not have $n + m$ maximal elements with at least n nuclear.*

Esakia duality and beyond

Esakia duality is an order-topological representation of Heyting algebras and their homomorphisms.

G. Bezhanishvili and Ghilardi (2007) extended Esakia duality for nuclear algebras. The extended Esakia spaces are called nuclear spaces.

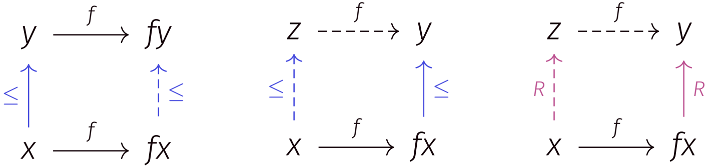
G. Bezhanishvili and N. Bezhanishvili (2009) generalised Esakia duality to account for $\{\wedge, \rightarrow\}$-homomorphisms between Heyting algebras.

To account for steady canonical formulas we need to extend this duality to the nuclear case.

Steady canonical formulas dually

A steady morphism is a partial Esakia morphism $f : X \to Y$ such that:

- $x \in \mathrm{dom}(f)$ implies $f[{\uparrow}x] = {\uparrow}fx$ and $R[fx] \subseteq f[R[x]]$



- $f[{\uparrow}x] = {\uparrow}y$ for some $y \in Y$ implies $x \in \mathrm{dom}(f)$
- $f[{\uparrow}x]$ is closed for all $x \in X$
- $X \setminus {\downarrow}f^{-1}(Y \setminus U)$ is clopen for each clopen upset $U \subseteq Y$.

Steady subframes

In the intermediate setting finite domains of onto partial Esakia morphisms are subframes.

Therefore, subframe logics are closed under finite domains of onto partial Esakia morphisms (finite domain property).

Steady logics have the finite domain property for steady morphisms.

In this sense steady logics are subframe lax logics.

Steady subframes

Definition – Steady subframes

*$(Y, \leq_Y, R_Y)$ is a **steady subframe** of $(X, \leq_X, R_X)$ iff*

- *$(Y, \leq_Y)$ is a subframe of $(X, \leq_X)$,*
- *R_Y is the largest lax relation contained in $R_X \cap Y^2$.*

Theorem – Steady subframe logics

A lax logic is axiomatized by steady formulas iff it is generated by a class closed under steady subframes.

Canonical formulas at work

Theorem

Suppose an intermediate logic $L' = IPC \oplus \Gamma$ has one of the properties: fmp, Kripke completeness, tabularity, or decidability. Then the lax logic $L = PLL \oplus \Gamma$ also has the same property.

Strategy:

1. For each $L \nvdash \phi$ find some $L' \nvdash \phi'$.
2. Transform an L'-frame $X' \nVdash \phi'$ into an L-frame $X \nVdash \phi$.

Above all, use (steady) canonical formulas!

Lax canonical formulas

Note, there is a strictly bigger reduct that we can use for lax logics:

Theorem – G. Bezhanishvili, N. Bezhanishvili, Carai, Gabelaia, Ghilardi, and Jibladze (2021)

The $\vee$-free reduct of nuclear algebras is locally finite.

However the associated canonical formulas are less flexible:

- Makes the proof for the previous theorem awkward.
- No simple axiomatization for steady logics.
- No finite domain property.
- Unlikely to be applicable to logics weaker than *PLL*.

Conclusion

The method of canonical formulas is a excellent tool to

1. Obtain classes of logics with the finite model property.
2. Prove preservation results.

The method is very exportable to the setting of intuitionistic modal logics with $\Box$.

Steady canonical formulas mirror stable canonical formulas and characterize lax logics in a flexible way.

Future work

More preservation results involving stable and steady canonical formulas.

Blok-Esakia theorems, e.g., embedding intuitionistic modal logics into bimodal logics.

Generally, investigating “semantic” translations.

Canonical formulas for multimodal logics and intuitionistic modal logic with $\Diamond$ as primitive.

Steady (canonical) formulas with the $(\Box, \to)$-free reduct.

Thank you!

Appendix: encoding function

Let $(A, D_\to, D_\Box)$ be a refutation pattern. For each $a \in A$ introduce a fresh propositional variable p_a.

Then define the stable canonical formula associated with $(A, D_\to, D_\Box)$ as $\alpha(A, D_\to, D_\Box) := \bigwedge \Box^+[\Gamma] \to \bigvee \Box^+[\Delta]$ where

$$\begin{aligned}\Gamma =& \{p_0 \leftrightarrow 0, p_1 \leftrightarrow 1\} \\ & \cup \{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid (a,b) \in A^2\} \\ & \cup \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \mid (a,b) \in A^2\} \\ & \cup \{p_{\Box a} \to \Box p_a \mid a \in A^2\} \\ & \cup \{p_{a\to b} \leftrightarrow p_a \to p_b \mid (a,b) \in D_\to\} \\ & \cup \{p_{\Box a} \leftrightarrow \Box p_a \mid a \in D_\Box\}\end{aligned}$$

and $\Delta = \{p_a \to p_b \mid (a,b) \in A^2 \ \& \ a \not\leq b\}$.

Appendix: encoding function

Note, stable canonical rules for IK would be given by Γ/Δ but to characterize these as formulas we need to use $IK4$-algebras.

Lemma
An IK-algebra is s.i. iff there exists an element $a \neq 1$ such that for any $b \neq 1$ there is a natural number k such that $\bigwedge_{i \leq k} \Box^i b \leq a$.

Corollary
An IK4-algebra is s.i. iff its $\Box^+$-fixed points have a second-largest element.

Nonetheless, the specific construction of such formulas is not important. We only have to make sure we get a fitting refutation criterion.

Appendix: refutation criterion

Lemma
$B \not\models \alpha(A, D_\to, D_\Box)$ iff there is a homomorphic image C of B and an stable bounded lattice embedding h from A into C which respects the parameters.

$$
\begin{aligned}
h\Box a &\leq \Box h a && \text{for all } a \in A \\
\Box h a &= h \Box a && \text{for all } a \in D_\Box \\
h(a \to b) &= ha \to hb && \text{for all } (a, b) \in D_\to .
\end{aligned}
$$

$$
B \twoheadrightarrow C \xhookleftarrow[D_\to D_\Box]{h} A
$$

This reason this is the desired refutation criterion lies in the locally finite reduct.

Appendix: stable filtration

That is, given $B \not\models \phi$ we can find an algebra A that refutes ϕ and is $|\mathrm{subf}(\phi)|$-generated as a bounded distributive lattice.

Let A be the bounded lattice generated by $v[\mathrm{subf}(\phi)] \subseteq B$.

Define the missing operations on A as

$$a \to_A b = \bigwedge \{c \in A \mid c \wedge a \leq b\}$$
$$\Box_A a = \bigvee \{\Box c \mid \Box c \in A \ \ \& \ \ \Box c \leq \Box a\}$$

Then $\Box_A a = \Box a$ whenever $\Box a \in v[\mathrm{subf}(\phi)]$ and $a \to_A b = a \to b$ whenever $a \to b \in v[\mathrm{subf}(\phi)]$.

Appendix: stable filtration

Then A is embeddable into B with a bounded lattice homomorphism h that respects the missing operations on (selected) elements of $v[\mathrm{subf}(\phi)]$.

However, the definition of the missing operations allows a stronger assumption on h. Namely,

$$h\Box a \leq \Box ha \qquad \text{for all } a, b \in A \qquad \text{(Stability)}$$

In other words, this “finitization” semi-preserves $\Box$.

Appendix: proof of theorem

1. For each $L \nvdash \phi$ find some $L' \nvdash \phi'$.

We can assume $\phi = \beta(A, D_\vee, D_\Box)$.

Besides, we have canonical formulas for int. logic, e.g.: $\beta'(A', D_\vee)$ for the Heyting reduct A' of A.

Moreover, we can show $L \nvdash \beta(A, D_\vee, D_\Box)$ implies $L' \nvdash \beta'(A', D_\vee)$.

Appendix: proof of theorem

2. Transform an L'-frame $X \nVdash \beta'(A', D_\vee)$ into an L-frame $X \nVdash \beta(A, D_\vee, D_\Box)$.

The trick for achieving this lies in S-spaces – an Esakia space $(X, \leq)$ with a subframe $S \subseteq X$.

G. Bezhanishvili and Ghilardi (2007) showed that nuclear spaces and S-spaces are in a one-to-one correspondence.

This means that we can define a nuclear frame from an int. Kripke frame by marking some subset.

$$X' \xhookrightarrow{h} (X'^*)_* \xhookleftarrow{g} C_* \overset{f}{\underset{D_\vee}{\twoheadrightarrow}} A_*$$

We use $h^{-1}[g[f^{-1}[S_A]]]$ to define a nuclear relation on X'.