Local compactness does not always imply spatiality

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May 19, 2025 – BLAST, Boulder

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Pointfree topology

The basic idea of pointfree topology is to study topological spaces without referring to points.

The popular approach is to study objects called frames (also known as locales):

Definition

A frame is a complete lattice in which finite meets distribute over arbitrary joins.

The motivating example is that for each topological space X, the lattice of open sets $\Omega(X)$ forms a frame. Frames of this form are called spatial.

McKinsey–Tarski algebras

Frame theory is not the only pointfree approach to topology.

Instead of abstracting lattices of the form $\Omega(X)$ to frames, one can abstract powerset lattices $\mathscr{P}(X)$ (equipped with their topological interior) to complete interior algebras.

This alternative, interior-based approach began with KURATOWSKI's closure axioms in 1922 and was further generalized by MCKINSEY and TARSKI in 1944.

Although it became central in modal logic, it was largely overlooked in pointfree topology, but recent work reintroduced McKinsey–Tarski (MT) algebras into the pointfree study of spaces.

MT-algebras

Definition

An MT-algebra M is a complete Boolean algebra equipped with an interior operator $\Box$.

For each topological space X, the powerset $\mathscr{P}(X)$ (equipped with the topological interior) forms an MT-algebra. Such MT-algebras are called spatial, and they are precisely the atomic MT-algebras.

For each MT-algebra $M = (M, \Box)$, the collection of open elements $\mathcal{O}(M) = \{a \in M \mid \Box a = a\}$ forms a frame.

Moreover, for each frame L there exists an MT-algebra M such that $\mathcal{O}(M) = L$ (the Funayama envelope $\mathcal{F}(L)$ of L). In this sense, the MT-algebra setting generalizes that of frames.

Separation axioms in pointfree topology

Separation axioms weaker than or equal to T_2 are infamously difficult (or even impossible) to describe in the setting of frames.

The category of spatial MT-algebras is dually equivalent to the category of topological spaces, allowing a pointfree generalization of all separation axioms.

Guram and Ranjitha formulated the separation axioms T_i for $i = 0, \frac{1}{2}, 1, 2, 3, 3\frac{1}{2}, 4$ in [GR, 2023]. This formulation is faithful: a space X is T_i iff $\mathscr{P}(X)$ is a T_i MT-algebra.

These definitions are also compatible with frame theory: under mild assumptions (e.g., T_1), M is T_i iff $\mathcal{O}(M)$ is T_i for $i = 3, 3\frac{1}{2}, 4$.

If M is T_2, then $\mathcal{O}(M)$ is T_2. The converse was an open question. As it turns out, the main counterexample in this talk also answers this in the negative.

Kubiak's comment

In a review of [GR, 2023], the following observation was made:

> "A coherent system of separation axioms [...] for MT-algebras is given by G. Nöbeling in his pioneering book[1] [...]. It is rather immediate (except possibly for $i = 2$) that an MT-algebra B is Nöbeling-T_i if and only if it is T_i ..."
>
> – TOMASZ KUBIAK

Despite being mentioned briefly by JOHNSTONE (1982) as an early example of pointfree topology, this remark points to a largely forgotten chapter in the history of pointfree topology. NÖBELING appears to be among the last to develop pointfree topology in the powerset-inspired direction, just before frame theory took over.

His work includes separation axioms, local compactness, and spatiality results in this setting.

[1] **Grundlagen der analytischen Topologie**, 1954.

Comparing separation axioms

Sparing you details, KUBIAK is correct that the definitions (except for T_2) are equivalent.

Lemma

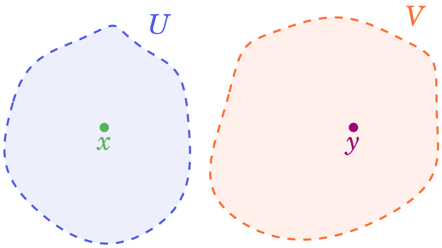
Let M be an MT-algebra. For $i = 1, 3, 3\frac{1}{2}, 4$, M is T_i iff M is Nöbeling-T_i.

Note that NÖBELING did not consider T_0.

For T_2, however, the definitions diverge.

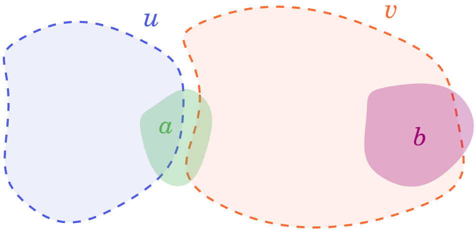
The classical T_2 condition

$$x \neq y \implies \exists U, V \in \Omega(X) \text{ with } x \in U,\ y \in V, \text{ and } U \cap V = \varnothing.$$



The Nöbeling T_2 condition

$$a \neq 0,\ b \neq 0, \text{ and } a \wedge b = 0 \implies \exists u, v \in \mathcal{O}(M) \text{ with } u \wedge a \neq 0,\ v \wedge b \neq 0, \text{ and } u \wedge v = 0.$$

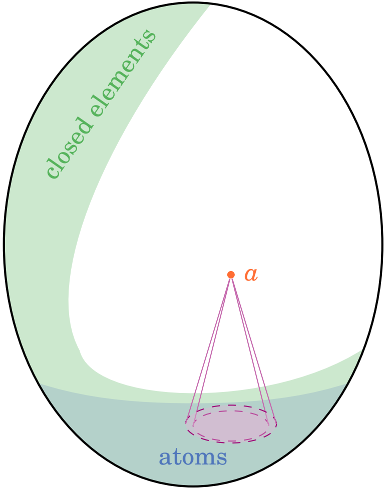


MT-style thinking about separation

In a topological space, every subset is a union of singletons. Analogously, in an atomic Boolean algebra, every element is a join of atoms.

This perspective allows us to describe separation axioms in terms of properties of a join-generating set.

- In a T_1 space, singletons are closed.
- In the pointfree world, this means: the closed elements join-generate the algebra.



Comparing Hausdorff

The T_2 property corresponds to a stronger approximation: a space X is Hausdorff iff

$$\{x\} = \bigcap\{\overline{U} \mid x \in U \in \Omega(X)\} \qquad \text{for every } x \in X.$$

Thus, in the MT setting, T_2 can be expressed as: every element is a join of elements of the form

$$c = \bigwedge\{\Diamond u \mid c \le u \in \mathcal{O}(M)\}. \qquad (\Diamond = \neg\Box\neg)$$

Theorem

1. *If M is T_2, then M is Nöbeling-T_2.*
2. *There exist Nöbeling-T_2 MT-algebras that are not T_1 (and hence not T_2).*

Example

Let B be a complete atomless Boolean algebra. Put $M = B \times B$ with $\Box(a,b) = (a \wedge b, b)$.

Comparing compactness

GURAM and RANJITHA studied compactness and local compactness of MT-algebras in [GR, 2025].

Once again, a comparison can be made with NÖBELING's corresponding notions.

Lemma

Let M be an MT-algebra.

1. *M is compact iff M is Nöbeling-compact.*
2. *If M is compact, then M is Nöbeling-locally compact.*
3. *If M is locally compact, then M is Nöbeling-locally compact.*
4. *If M is T_2, then M is Nöbeling-locally compact iff M is locally compact.*

The fact that Nöbeling-locally compact and locally compact are very different is unsurprising.

Let X be a topological space. Most commonly X is called **locally compact** if every point x of X has a compact neighbourhood, i.e., there exists an open set U and a compact set K, such that $x \in U \subseteq K$.

There are other common definitions: They are all **equivalent if X is a Hausdorff space** (or preregular). But they are **not equivalent** in general:

Nöbeling-locally compact — 1. every point of X has a compact neighbourhood.

2. every point of X has a closed compact neighbourhood.

2′. every point of X has a relatively compact neighbourhood.

2″. every point of X has a local base of relatively compact neighbourhoods.

Locally compact — 3. every point of X has a local base of compact neighbourhoods.

4. every point of X has a local base of closed compact neighbourhoods.

5. X is Hausdorff and satisfies any (or equivalently, all) of the previous conditions.

Logical relations among the conditions:[2]

- Each condition implies (1).
- Conditions (2), (2′), (2″) are equivalent.
- Neither of conditions (2), (3) implies the other.
- Condition (4) implies (2) and (3).
- Compactness implies conditions (1) and (2), but not (3) or (4).

In short, they generalize different properties.

Spatiality theorems

Nöbeling proved several spatiality theorems. One of them is:

Theorem (Nöbeling, 1954)

Compact T_1 MT-algebras are spatial.

There is a connection between T_1 MT-algebras and subfit frames, which allows one to derive Isbell's Spatiality Theorem from Nöbeling's result.

Corollary (Isbell, 1972)

Compact subfit frames are spatial.

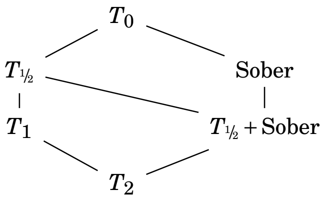
Nöbeling-locally compact T_1 algebras are spatial

NÖBELING also generalized his spatiality result to the Nöbeling-locally compact case:

Theorem (Nöbeling, 1954)

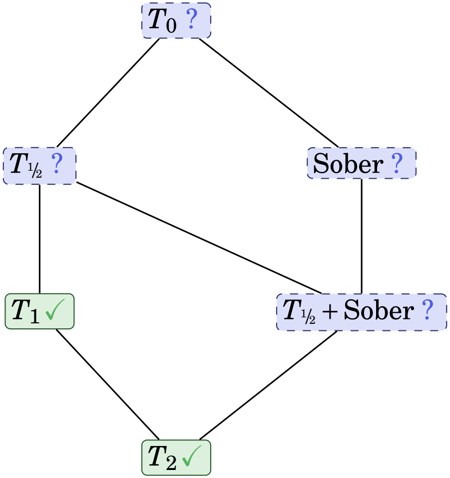
Nöbeling-locally compact T_1 MT-algebras are spatial.

He did not consider weaker separation axioms such as T_0, $T_{1/2}$, or sobriety.



This leads to a natural question: how much separation is actually needed to make Nöbeling-locally compact MT-algebras spatial?

Which Nöbeling-locally compact MT-algebras are spatial?



Nöbeling-locally compact sober $T_{1/2}$ algebras are not spatial

The search for a weaker separation axiom that guarantees spatiality under Nöbeling-local compactness ends early:

Theorem

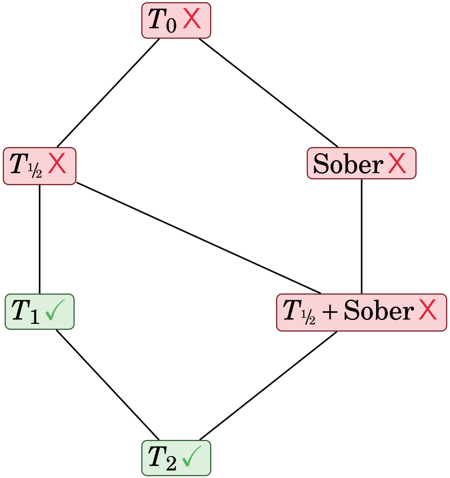
There exist compact (and hence Nöbeling-locally compact) sober $T_{1/2}$ MT-algebras that are not spatial.

Example

Let L be a complete atomless Boolean algebra with an extra top element adjoined. Then $\mathcal{F}(L)$ is compact, sober, and $T_{1/2}$, but has only one atom.

This shows that Nöbeling-local compactness does not provide enough local information to ensure spatiality.

Which Nöbeling-locally compact MT-algebras are spatial?



Nöbeling-local compactness is not a local property

In a T_0 MT-algebra, compact elements must contain atoms:

Lemma

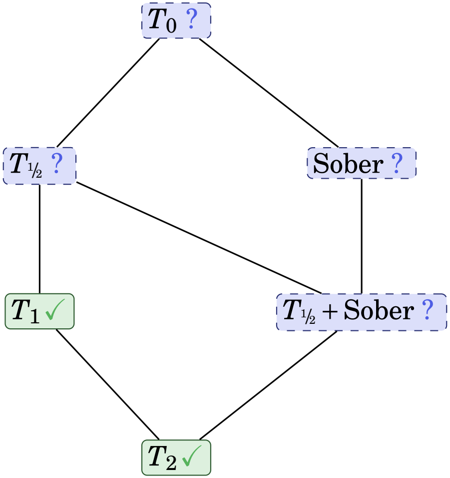
Let M be a T_0 MT-algebra. If $k \in M$ is nonzero and compact, then there exists an atom $x \in M$ such that $x \leq k$.

However, if there are too few compact elements, this does not generate enough atoms to make the algebra atomic.

In fact, a compact (and hence Nöbeling-locally compact) MT-algebra may contain only a single compact element (recall the the previous example).

We will see that this changes when working with locally compact MT-algebras.

Which locally compact MT-algebras are spatial?



The $T_{1/2}$ + sober case

When working with sober $T_{1/2}$ MT-algebras, spatiality can be recovered from the frame:

Theorem (Guram–Ranjitha, 2023)

If M is sober and $T_{1/2}$, then M is spatial iff $\mathcal{O}(M)$ is spatial.

Moreover, frame-theoretically nothing* is required: (*Prime Ideal Theorem)

Theorem (Hofmann–Lawson, 1977)

Continuous frames are spatial.

(Continuity is the frame-theoretic analogue of local compactness.)

Combining the two gives:

Corollary

Locally compact sober $T_{1/2}$ MT-algebras are spatial.

The $T_{1/2}$ case

We can in fact do better: sobriety is not necessary. Recall the following fact:

Lemma

Let M be a T_0 MT-algebra. If $k \in M$ is nonzero and compact, then there exists an atom $x \in M$ such that $x \leq k$.

In the locally compact $T_{1/2}$ setting, we can localize this lemma to every nonzero element:

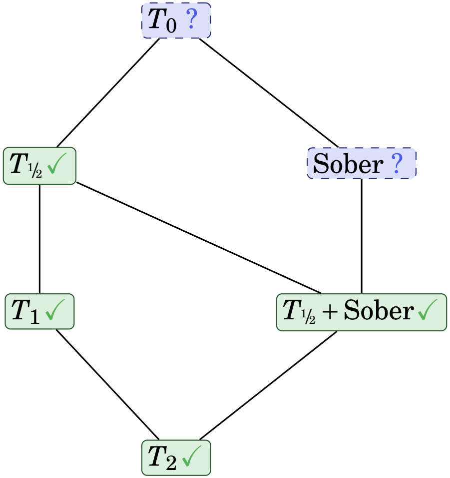
Theorem

If M is locally compact and $T_{1/2}$, then below every nonzero element there exists a nonzero compact element.

Corollary

Locally compact $T_{1/2}$ MT-algebras are spatial.

Which locally compact MT-algebras are spatial?



The sober case and Raney extensions

The spatiality of locally compact sober MT-algebras is the main open problem in this talk.

Since such algebras are spatial when $T_{1/2}$ holds, any counterexample must fail $T_{1/2}$. Examples like this cannot come from Funayama envelopes of frames as those are always $T_{1/2}$ (see the next talk).

Instead, we turn to Raney extensions, as studied by ANNA.

Roughly speaking, just as frames correspond to lattices of open sets, Raney extensions correspond to lattices of saturated sets.

Raney extensions and T_0

Raney extensions play the same role for T_0 MT-algebras as frames do for $T_{1/2}$ MT-algebras.

For any MT-algebra M, the lattice of saturated elements $\mathcal{S}at(M)$ forms a Raney extension. Conversely, for any Raney extension C, the Funayama envelope $\mathcal{F}(C)$ is a T_0 MT-algebra.

We also have:

Lemma

Let C be a Raney extension.

1. *C is spatial iff $\mathcal{F}(C)$ is spatial.*
2. *C is sober iff $\mathcal{F}(C)$ is sober.*

The main example

Every frame has a largest and smallest Raney extension.

Of particular interest is the largest Raney extension C of the frame $\Omega(\mathbb{R})$, where $\mathbb{R}$ carries the standard topology.

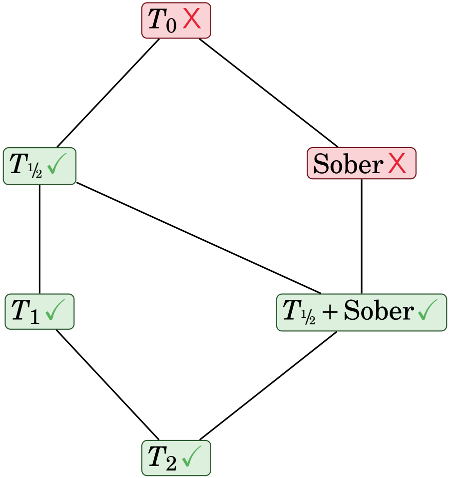
In this case, C is sober but not spatial. Therefore, $\mathscr{F}(C)$ is sober and not spatial.

Moreover, since $\Omega(\mathbb{R})$ is continuous and $\mathscr{F}(C)$ is sober, $\mathscr{F}(C)$ is locally compact.

Theorem

There exist locally compact sober MT-algebras that are not spatial.

Which locally compact MT-algebras are spatial?



Conclusion

Frames are blind to weak separation. They always seem $T_{1/2}$, so local compactness always makes them spatial.

MT-algebras aren't blind. They see more, and sometimes, that means seeing too little separation to be spatial.

Local compactness does not always imply spatiality

Thank you!