Pointfree topology and Priestley duality

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- Avoids reliance on strong assumptions (e.g., Axiom of Choice).
- Provides an algebraic perspective on topology.

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How can we bridge this gap?

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Restricts to a dual equivalence between the category of frames and a subcategory of Priestley spaces.

Provides a tool to study spatial as well as non-spatial frames using order and topology.

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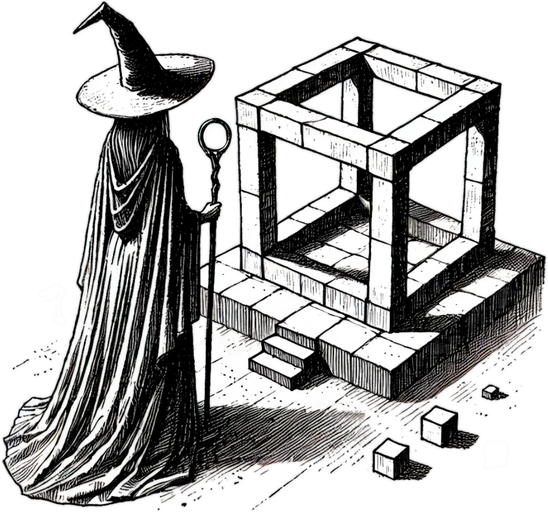
- Explore the role of Priestley duality in pointfree topology.
- Demonstrate how Priestley duality provides a fresh perspective on classic results in pointfree topology.
- Understand frames through their Priestley spaces.



Foundations of Priestley duality for frames

Frames

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1. A frame (or locale) is a complete lattice L satisfying

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For each topological space X, the lattice of opens $\Omega(X)$ is a frame, and for each continuous map $f : X \to Y$, the inverse image $f^{-1} : \Omega(Y) \to \Omega(X)$ is a frame homomorphism.

Adjunction

There is a well-known dual adjunction

$$\mathsf{Top} \overset{\Omega}{\rightleftarrows} \mathsf{Frm}$$

where

- Top is the category of topological spaces and continuous maps, and
- Frm is the category of frames and frame homomorphisms.

Adjunction

There is a well-known dual adjunction

$$\mathsf{Top} \underset{\mathrm{pt}}{\overset{\Omega}{\rightleftarrows}} \mathsf{Frm}$$

where

- Top is the category of topological spaces and continuous maps, and
- Frm is the category of frames and frame homomorphisms.
- pt maps frames to their spaces of points (completely prime filters—a filter F such that $\bigvee S \in F$ implies $S \cap F \neq \varnothing$).

The units of this adjunction are

- $\lambda : X \to \mathrm{pt}(\Omega(X))$ defined by $\lambda(x) = \{U \in \Omega(X) \mid x \in U\}$.
- $\varphi : L \to \Omega(\mathrm{pt}(L))$ defined by $\varphi(a) = \{x \in \mathrm{pt}(L) \mid a \in x\}$.

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φ is an isomorphism iff $a \not\leq b$ implies there exists $x \in \mathrm{pt}(L)$ such that $a \in x \not\ni b$. We call such frames spatial.

The adjunction restricts to a dual equivalence between:

- Sob the full subcategory of Top consisting of sober spaces
- SFrm the full subcategory of Frm consisting of spatial frames

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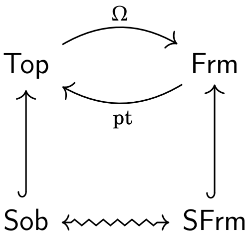

where

$\leftrightsquigarrow$ represents a dual equivalence

$\hookrightarrow$ represents a full subcategory

Priestley duality

A Priestley space is a compact topological space X equipped with a partial order $\leq$ satisfying

$$x \not\leq y \implies \exists U \in \mathrm{ClopUp}(X) : x \in U \text{ and } y \notin U$$

$$(\mathrm{ClopUp}(X) = \text{clopen upsets of } X)$$

for all $x, y \in X$.

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A Priestley morphism is a continuous order-preserving map between Priestley spaces.

Let Pries be the category of Priestley spaces and Priestley morphisms and DLat the category of bounded distributive lattices and bounded lattice homomorphisms.

Theorem (Priestley, 1970)

Pries *and* DLat *are dually equivalent.*

Stone duality

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Theorem (Stone, 1936)

Stone *and* BA *are dually equivalent.*

BA is a full subcategory of DLat, and Stone is a full subcategory of Pries. Thus, Priestley duality generalizes Stone duality:

$$\begin{array}{ccc} \mathsf{DLat} & \leftrightsquigarrow & \mathsf{Pries} \\ \uparrow & & \uparrow \\ \mathsf{BA} & \leftrightsquigarrow & \mathsf{Stone} \end{array}$$

Pultr–Sichler duality

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Definition

1. An L-space (or localic space) is a Priestley space X satisfying

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3. Let LPries be the category of L-spaces and L-morphisms.

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Key considerations:

- The correspondence between bounded lattice morphisms and Priestley morphisms restricts appropriately to frame homomorphisms and L-morphisms.
- Isomorphisms in Pries (or DLat) between L-spaces (or frames) remain isomorphisms in LPries (or Frm).

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Proposition

$\mathrm{pt}(L)$ *is homeomorphic to* $\mathrm{loc}X$.

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Corollary

SLPries *and* SFrm *are dually equivalent.*

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Conversely, we define a functor $\mathscr{P}r\colon \mathsf{Top} \to \mathsf{LPries}$ by mapping a topological space X to the Priestley space of its frame of opens.

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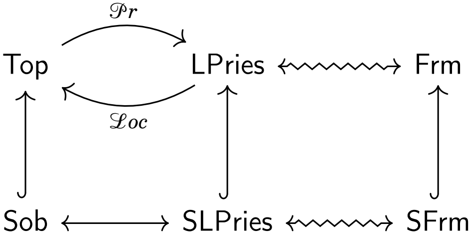
These functors restrict to an equivalence:

Theorem

SLPries *and* Sob *are equivalent.*

We thus obtain a new perspective on the classical adjunction of pointfree topology:



Hofmann–Mislove through the lenses of Priestley

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It states that for each sober space Z, the Scott-open filters of the frame $\Omega(Z)$ are (dually) isomorphic to the compact saturated subsets of Z.

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If X is the Priestley space of $\Omega(Z)$, then $\Omega(Z) \cong \mathrm{ClopUp}(X)$. Thus, Priestley's result restricts to Scott-open filters and special closed upsets of the Priestley space.

Scott upsets

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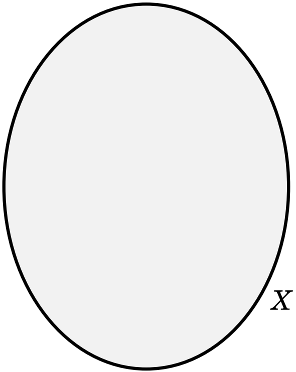

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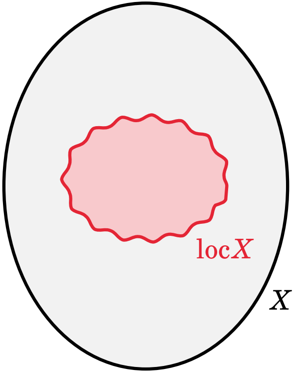

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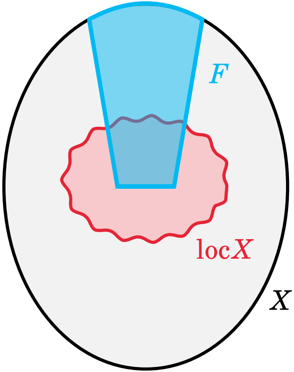

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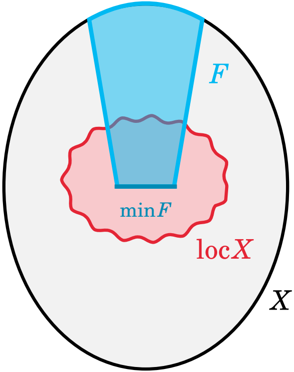

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Theorem

Let L be a frame and X its Priestley space. For a filter $F \subseteq L$ and its dual closed upset $K \subseteq X$, the following are equivalent:

1. *F is Scott-open.*
2. *If $U \subseteq X$ is an open upset, then $K \subseteq \mathrm{cl}\, U$ implies $K \subseteq U$.*
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Corollary

The poset $\operatorname{OFilt}(L)$ of Scott-open filters of a frame L is dually isomorphic to the poset of $\operatorname{SUp}(X)$ of Scott upsets of its Priestley space X.

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Corollary (Hofmann–Mislove)

1. *Let L be a frame. Then* $\mathrm{OFilt}(L)$ *is dually isomorphic to* $\mathrm{KSat}(\mathrm{pt}(L))$.
2. *Let X be a sober space. Then* $\mathrm{OFilt}(\Omega(X))$ *is dually isomorphic to* $\mathrm{KSat}(X)$.

The way-below relation and compactness

In a frame L, we define the way-below relation $\ll$ by

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Proposition

Let L be a frame and X its Priestley space.

1. *$a \ll b$ iff $\varphi(a) \ll \varphi(b)$.*
2. *a is compact iff $\varphi(a)$ is a Scott upset.*
3. *L is compact iff $\min X \subseteq \mathrm{loc} X$.*

Compactness

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Consider the following categories:

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Theorem

1. KLPries *is dually equivalent to* KFrm.
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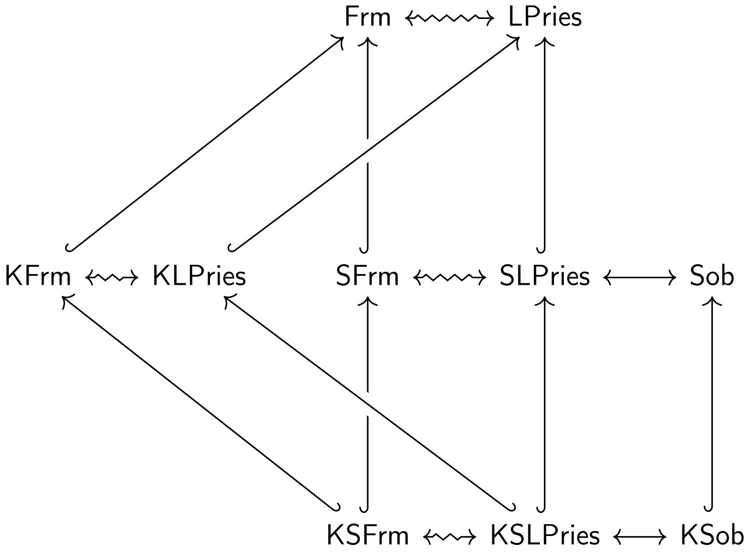
Frm
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A Priestley journey from Hofmann–Lawson to Isbell

Kernels

We now characterize various important classes of frames via kernels.

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Definition

Let X be an L-space.

1. A map $\mathtt{ker}\colon \mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$ is a kernel if
   - $\mathtt{ker}\, U \subseteq U$ and
   - $U \subseteq V$ implies $\mathtt{ker}\, U \subseteq \mathtt{ker}\, V$

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One of the main contributions of this work is identifying the appropriate kernels that correspond to key frame properties.

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A frame L is continuous if $a = \bigvee\{b \in L \mid b \ll a\}$.

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Theorem

A frame is continuous iff con *is representative in its Priestley space.*

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A continuous frame L is stably continuous if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$.

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Theorem

Let L be a frame and X its Priestley space

1. *L is stably continuous iff* `con` *is representative and stable.*
2. *L is stably compact iff* `con` *is representative, stable, and top-preserving.*

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We characterize the Priestley spaces of these classes of frames via a new kernel.

Definition

Let X be an L-space. The algebraic kernel is defined by

$$\mathtt{alg}\, U = \bigcup \{V \in \mathrm{ClopSUp}(X) \mid V \subseteq U\}.$$

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Theorem

Let L be a frame and X its Priestley space.

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Regularity and zero-dimensionality

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Definition

Define the following kernels:

1. regular kernel: $\mathtt{reg}\, U = \bigcup\{V \in \mathrm{ClopUp}(X) \mid {\downarrow} V \subseteq U\}$
2. zero-dimensional kernel: $\mathtt{zer}\, U = \bigcup\{V \in \mathrm{ClopBi}(X) \mid V \subseteq U\}$

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Note that $\mathtt{reg}$ and $\mathtt{zer}$ are always stable and top-preserving.

Comparison of kernels

Recall, every algebraic frame is continuous and every regular frame is zero-dimensional.

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Lemma

$\mathtt{alg} \leq \mathtt{con}$ *and* $\mathtt{zer} \leq \mathtt{reg}$.

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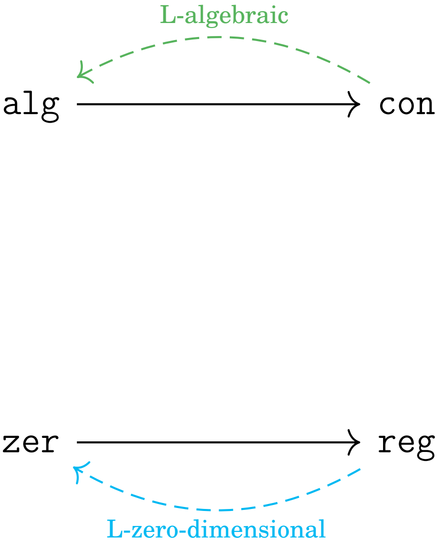
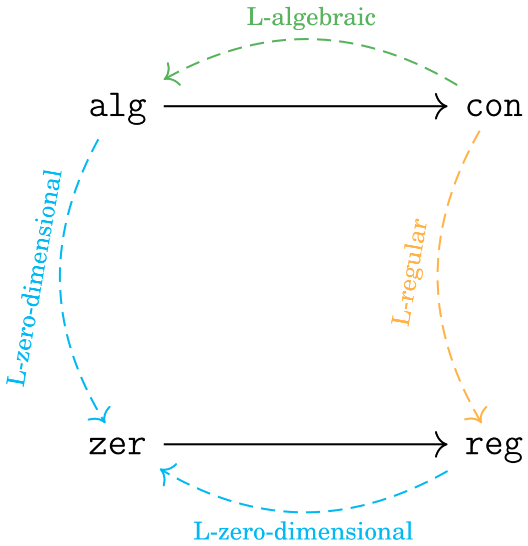
Lemma

1. *X is the Priestley space of a compact frame iff* $\mathtt{reg} \leq \mathtt{con}$.
2. *X is the Priestley space of a compact regular frame iff* $\mathtt{reg} = \mathtt{con}$ ($\Leftarrow$*-direction requires X to be* L*-spatial*).

`alg` $\longrightarrow$ `con`

`zer` $\longrightarrow$ `reg`


L-algebraic
alg
con
zer
reg
L-zero-dimensional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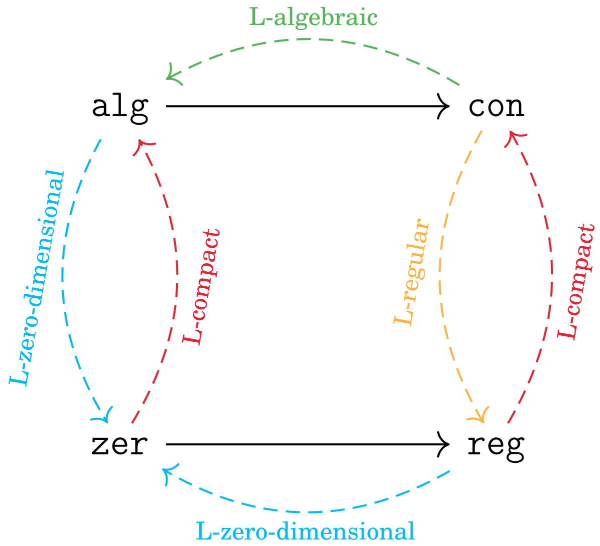

L-algebraic
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Deriving classic dualities

Using our characterizations we can define various subcategories of LPries.

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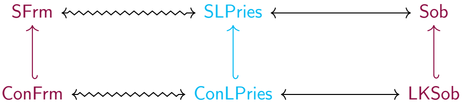
- Hofmann–Lawson duality between the categories ConFrm of continuous frames and LKSob of locally compact sober spaces.
- Isbell duality between the categories KRFrm of compact regular frames and KHaus of compact Hausdorff spaces.

$$\mathsf{SFrm} \leftrightsquigarrow \mathsf{SLPries} \longleftrightarrow \mathsf{Sob}$$


SFrm
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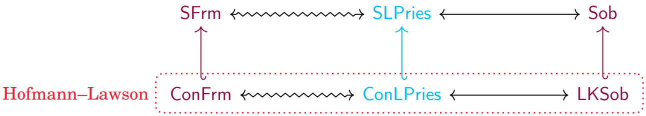
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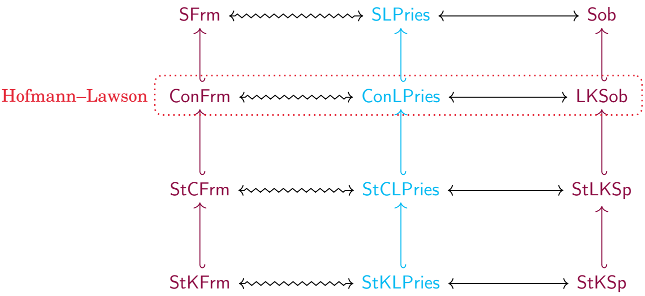
SFrm
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StCFrm
StCLPries
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StKFrm
StKLPries
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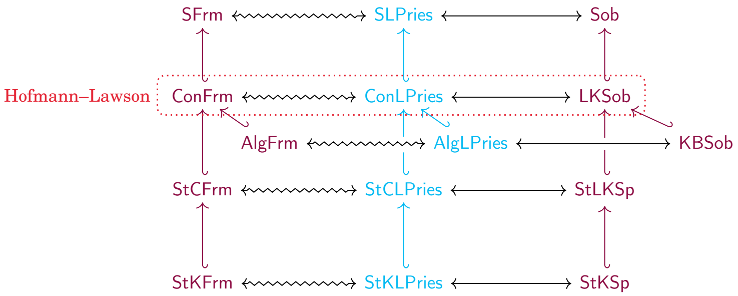
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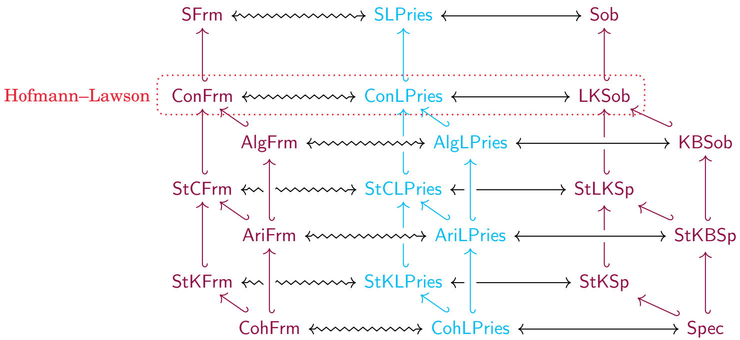
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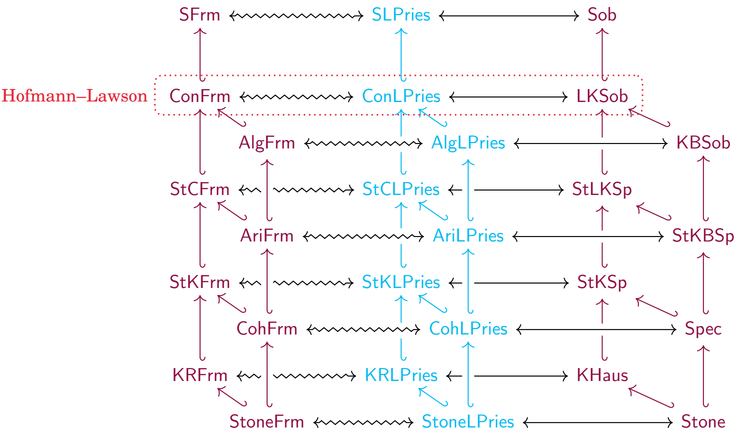
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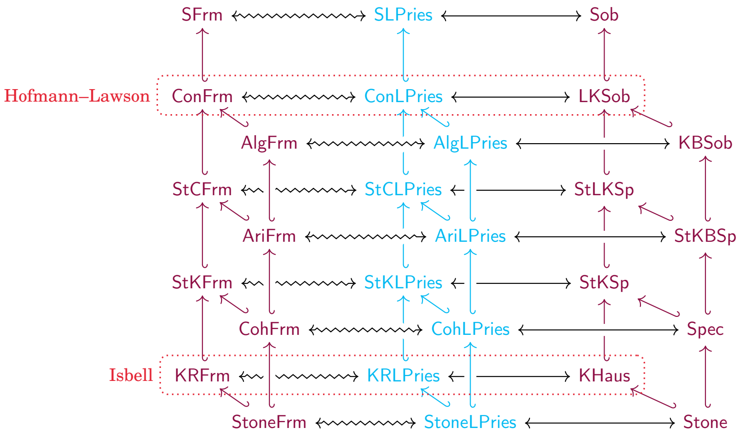
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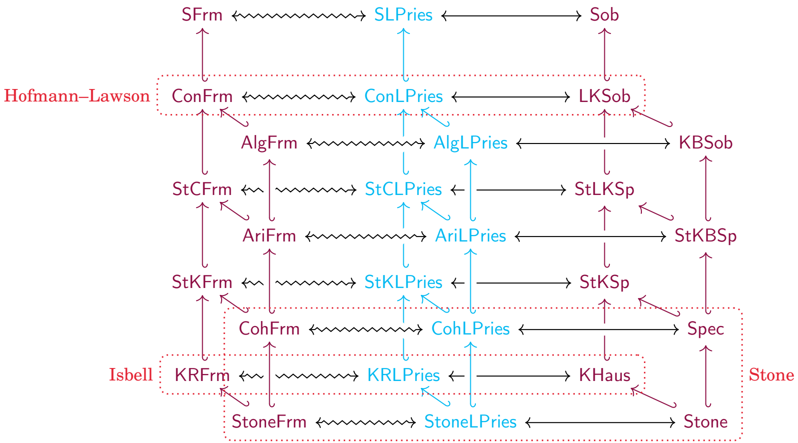
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Future work

Conclusion

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The newly introduced categories of Priestley spaces correspond not only to significant classes of frames but also to key categories of topological spaces, including compact Hausdorff spaces and Stone spaces.

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The newly introduced categories of Priestley spaces correspond not only to significant classes of frames but also to key categories of topological spaces, including compact Hausdorff spaces and Stone spaces.

This perspective offers new insights into classic duality results in pointfree topology.

Spectra of maximal d-elements

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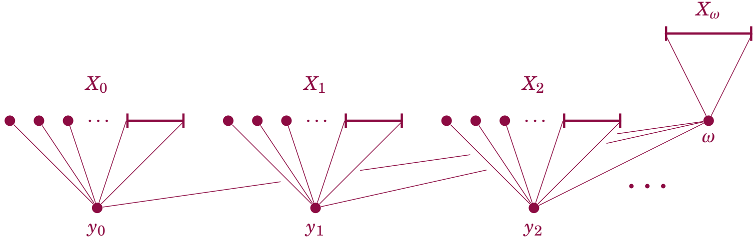

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This direction is currently under investigation, and using this Priestley framework we expect further insights into this and other spectra in pointfree topology in the future.

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A solution would shed some light on Isbell's problem.

Thank you!