Priestley duality and its applications to pointfree topology

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November 11, 2024 – (Mexico City, MITAC)

Pointfree Topology

One of the key ideas in pointfree topology is to study topological spaces in terms of their lattices of open sets, rather than relying on their points.

This approach enables reasoning about topological properties without reference to points (hence the name "pointfree").

However, despite generalizing many topological concepts, pointfree topology lacks certain intuitive aspects tied to point-based structures.

Goal of this course: to introduce a framework in which every frame can be faithfully represented as a topological space, reconnecting frames with point-based intuition.

Structure of the course

- Background
  - Pointfree topology
  - Stone duality and spectral spaces
- Priestley duality for frames
  - Priestley spaces
  - Esakia duality
  - Pultr-Sichler duality
- Pointfree topology in the language of Priestley spaces
  - Spatiality
  - Compactness
  - Continuity and regularity
  - Various dualities

Background:

Pointfree topology

A little history

Geometry has always been central to mathematics, but it was only in the 19th century, when the rise of modern analysis, that precise notions of continuity, convergence, etc. became essential.

Hausdorff introduced the concept of a *neighborhood*, shifting focus from a distance-based definitions to spatial relationships.

The modern view of topology further developed through the work of Kuratowski, Alexandroff, Ursyohn, and Sierpiński.

A little history

From the 1930s onward, the idea of representing spaces through lattices of open sets gained traction.

Contributions from Stone, Wallman, McKinsey & Tarski, and others in the 1930s and 1940s laid early foundations for a pointfree approach.

Research in Ehresmann's seminar formally introduced frames as pointfree spaces.

Following this, Dowker & Papert, Isbell, Banaschewski, and others expanded the field rapidly.

Johnstone's book *Stone Spaces* solidified pointfree topology's significance, while Picado & Pultr further advanced the field with *Frames and Locales* and *Separation in Point-Free Topology*.

Frames

For every topological space X, the lattice $\mathcal{O}(X)$ of open sets forms a frame – a complete lattice where arbitrary joins commute with meet:

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

A frame homomorphism is a map between frames that preserves arbitrary joins and finite meets. If $f : X \to Y$ is a continuous map, then the inverse image $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a frame homomorphism.

This assignments yields a contravariant functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Frm}$, where

Frm – the category of frames and frame homomorphisms

Top – the category of topological spaces and continuous maps.

Points of a frame

Not every frame L can be represented as $\mathcal{O}(X)$ for some topological space; in other words, not all frames are *spatial*.

However, for every frame L we can associate a topological space of points.

Lemma

There is a one-to-one correspondence between:

1. *frame homomorphisms* $h : L \to 2$*,*
2. *meet-irreducible elements of* L*, and*
3. *completely prime filters of* L*.*

Each of these correspond to a point of L.

The space of points

For a frame L, we define $\mathcal{pt}(L)$, the space of points of L, as the collection of completely prime filters, topologized by open sets of the form:

$$\varphi(a) = \{x \in \mathcal{pt}(L) \mid a \in x\}, \qquad a \in L$$

For each frame homomorphism $h : L \to M$, the inverse image gives a continuous map $h^{-1} : \mathcal{pt}(M) \to \mathcal{pt}(L)$.

This assignment defines a contravariant functor $\mathcal{pt} : \mathbf{Frm} \to \mathbf{Top}$.

The adjunction

We have the following adjunction between categories:

$$\mathbf{Top} \underset{pt}{\overset{\mathcal{O}}{\rightleftarrows}} \mathbf{Frm}$$

The units of this adjunction are:

- $\lambda : X \to pt(\mathcal{O}(X))$, defined by $\lambda(x) = \{U \in \mathcal{O}(X) \mid x \in U\}$ — in other words, $\lambda(x)$ is the completely prime filter of open sets containing x.
- $\varphi : L \to \mathcal{O}(pt(L))$, as defined earlier.

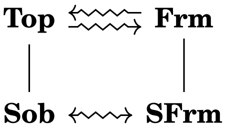
The equivalence

The map λ is an homeomorphism iff each completely prime filter of $\mathcal{O}(X)$ has the form $\lambda(x)$ for some unique $x \in X$. Spaces with this property are called sober.

φ is an isomorphism iff $a \not\leq b$ implies there exists $x \in pt(L)$ such that $a \in x$ but $b \notin x$. Frames with this property are called spatial.

Thus, the adjunction restricts to an equivalence between:
Sob – full subcategory of sober spaces

SFrm – full subcategory of spatial frames



where

$\overset{\leftsquigarrow}{\rightsquigarrow}$ represents a dual adjunction

$\leftrightsquigarrow$ represents a dual equivalence

$\backslash$ represents a subcategory

Pointless frames

This means we can use the functor pt to represent frames as a topological spaces.

However, pt can lose significant information about frames.

Example

For a boolean algebra, there is a one-to-one correspondence between atoms and points. Thus, if L is a a complete atomless boolean algebra then $pt(L) = \varnothing$.

In other words, L is a pointless frame.

How to interpret a “space” without points?

Locales

In short, pt provides a faithful representation only for spatial frames.

A common alternative is to consider the opposite category $\mathbf{Frm}^{\mathrm{op}}$ as a generalization of sober topological spaces. In this approach, frames are referred to as locales.

This functor $\mathbf{Frm} \to \mathbf{Frm}^{\mathrm{op}}$ retains all information, but what sense of geometric intuition do locales retain, comparable to topological spaces?

We'll explore an alternative approach by examining Stone duality.

Background:

Stone duality and spectral spaces

Stone Duality for Boolean Algebras

Stone pursued questions about which algebraic structures can be represented as topological spaces that ultimately led to his celebrated duality for Boolean algebras.

A Stone space is defined as a zero-dimensional compact Hausdorff space.

Stone — the category of Stone spaces and continuous maps

BA — the category of Boolean algebras and Boolean homomorphisms

Theorem (Stone, 1936)

BA *and* **Stone** *are dually equivalent.*

Stone Duality for Distributive Lattices

Following this, Stone extended his duality to bounded distributive lattices. The corresponding spaces, described with some complexity, are now known as spectral spaces:

Definition (Spectral Spaces)

A spectral space is a topological space X that is sober and coherent — meaning the compact open sets form a basis that is a bounded sublattice of $\mathcal{O}(X)$.
A spectral map is a continuous map that pull compact open sets back to compact open sets.

The complexity may be one reason why this duality is not as widely known as Stone's Boolean duality.

Stone Duality for Distributive Lattices

Spec — the category of spectral spaces and spectral maps

DLat — the category of bounded distributive lattices and their homomorphisms

Theorem (Stone, 1938)

DLat *and* **Spec** *are dually equivalent.*

This yields the following diagram:

$$\begin{array}{ccc} \mathbf{DLat} & \leftrightsquigarrow & \mathbf{Spec} \\ | & & | \\ \mathbf{BA} & \leftrightsquigarrow & \mathbf{Stone} \end{array}$$

Spectral spaces as locales

Since frames are bounded distributive lattices satisfying an additional condition, and frame homomorphisms are specific bounded lattice homomorphisms, we can refine Stone duality for distributive lattices to frames.

$$\begin{array}{ccc} \mathbf{DLat} & \leftrightsquigarrow & \mathbf{Spec} \\ | & & | \\ \mathbf{Frm} & \leftrightsquigarrow & \mathbf{?} \end{array}$$

This yields a category of spectral spaces and spectral maps that faithfully represents all frames, including both spatial and non-spatial frames.

This approach has been explored in work by Schwartz (2013) and is further detailed in the book *Spectral Spaces* (2019).

Priestley duality for frames:

Priestley spaces

Priestley spaces

In this course, we will consider a different approach.

In the 1970s, Priestley developed a duality for bounded distributive lattices using compact spaces equipped with a continuous partial order. These spaces are now called Priestley spaces.

Priestley spaces have since become a powerful and intuitive tool in studying distributive lattices and their subcategories.

Esakia further developed a duality for Heyting algebras by imposing additional restrictions.

Pultr & Sichler initiated the investigation of Priestley spaces of frames, a topic that has since seen considerable development and is the subject of this workshop.

Priestley spaces

Definition

- A Priestley space is a compact space with a partial order $\leq$ satisfying the Priestley separation axiom:

  whenever $a \not\leq b$ there exists a clopen upset U such that
  $$a \in U \not\ni b.$$

  (clopen = closed and open)
- A Priestley morphism is an continuous, order-preserving map between Priestley spaces.

Pries – Priestley spaces and Priestley morphisms

Theorem (Priestley, 1970 & 1972)

DLat *and* **Pries** *are dually equivalent.*

From distributive lattice to Priestley space.

To see how Priestley duality is established, let's examine the functors involved.

For $D \in \mathbf{DLat}$, let $\mathcal{X}(D)$ be the collection of prime filters of D topologized by the subbasis $\{\varphi(a) \mid a \in D\} \cup \{\mathcal{X}(D) \setminus \varphi(b) \mid b \in D\}$.

where φ is the Stone map: $\varphi(a) = \{x \in \mathcal{X}(D) \mid a \in x\}$

Proposition

$(\mathcal{X}(D), \subseteq)$ is a Priestley space.

Proof.

Let $X = (\mathcal{X}(D), \subseteq)$. We need to show that X satisfies the Priestley separation axiom and that X is compact. Suppose $x, y \in X$ with $x \nsubseteq y$. Then there exists $a \in x$ such that $a \notin y$, meaning $x \in \varphi(a)$ and $y \notin \varphi(a)$. Since $\varphi(a)$ is a clopen upset, the Priestley separation axiom holds.

Proof continued.

To show compactness, we apply Alexander Subbase Theorem. So assume $X = \bigcup_i \varphi(a_i) \cup \bigcup_j X \setminus \varphi(b_j)$. Let F be the filter generated by $\{b_j\}$ and I the ideal generated by $\{a_i\}$.

If $F \cap I = \varnothing$, then the Prime Filter Theorem allows us to find $x \in X$ with $F \subseteq x$ and $I \cap x = \varnothing$. By assumption, either $x \in \varphi(a_i)$ or $x \in X \setminus \varphi(b_j)$. If $x \in \varphi(a_i)$, then $a_i \in x$, giving $x \cap I \neq \varnothing$, a contradiction. Similarly, if $x \in X \setminus \varphi(b_j)$, then $b_j \notin x$, but $b_j \in F \subseteq x$, also a contradiction.

Hence, $F \cap I \neq \varnothing$, implying the existence of $c \in F \cap I$, which means $b_1 \wedge \cdots \wedge b_n \leq c \leq a_1 \vee \cdots \vee a_m$. Thus,

$$\varphi(b_1) \cap \cdots \cap \varphi(b_n) \subseteq \varphi(a_1) \cup \cdots \cup \varphi(a_m),$$

and so

$$X = \varphi(a_1) \cup \cdots \cup \varphi(a_m) \cup X \setminus \varphi(b_1) \cup \cdots \cup X \setminus \varphi(B_n),$$

proving that X is compact. □

From distributive lattice to Priestley space: morphisms

So for each $D \in \mathbf{DLat}$, we obtain a Priestley space $\mathcal{X}(D)$.

For a bounded lattice homomorphism $h : D \to D'$, we assign the map $\mathcal{X}(h) = h^{-1} : \mathcal{X}(D') \to \mathcal{X}(D)$. This is well defined since bounded lattice homomorphisms pull prime filters back to prime filters.

Lemma

h^{-1} is a Priestley morphism.

Proof sketch.
We need to show that it is order-preserving and continuous. Order-preservation follows since $x \subseteq y$ implies $h^{-1}(x) \subseteq h^{-1}(y)$. For continuity, we show $\varphi(h(a)) = h^{-1}(\varphi(a))$. □

From Priestley space to distributive lattice

For $X \in \mathbf{Pries}$, let $\mathscr{D}(X) = \mathsf{ClopUp}(X)$ be the collection of clopen upsets of X.

For a Priestley morphism $f : X \to Y$, let $\mathscr{D}(f) = f^{-1} : \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$.

The following results can be verified directly.

Lemma

$(\mathscr{D}(X), \cup, \cap, \varnothing, X)$ *is a bounded distributive lattice.*

Lemma

$\mathscr{D}(f)$ *is a bounded lattice homomorphism.*

Thus, we have contravariant functors $\mathscr{X} : \mathbf{DLat} \to \mathbf{Pries}$ and $\mathscr{D} : \mathbf{Pries} \to \mathbf{DLat}$.

$$\mathbf{Pries} \underset{\mathscr{X}}{\overset{\mathscr{D}}{\rightleftarrows}} \mathbf{DLat}$$

The units of this adjunction are:

- the Stone map $\varphi : D \to \mathscr{D}(\mathscr{X}(D))$, as defined earlier.
- $\lambda : X \to \mathscr{X}(\mathscr{D}(X))$, defined by $\lambda(x) = \{U \in \mathsf{ClopUp}(X) \mid x \in U\}$ — in other words, $\lambda(x)$ is the prime filter of clopen open sets containing x.

Lemma

φ is an isomorphism and λ is a homeomorphism.

The proof requires the Prime Ideal Theorem, but our hands are already dirty.

Priestley and Spectral

This leads us to the following diagram

$$\mathbf{Pries} \leftrightsquigarrow \mathbf{DLat} \leftrightsquigarrow \mathbf{Spec}$$

It is evident that **Pries** and **Spec** are equivalent.

In fact, they are not just equivalent but are actually isomorphic as categories.

This isomorphism was established by Cornish in 1975.

Let's explore this isomorphism further.

Upper topology

The collection of open upsets $\mathsf{OpUp}(X)$ in a Priestley space X form a topology.

Lemma

$\mathfrak{s}(X) = (X, \mathsf{OpUp}(X))$ *is a spectral space*

Proof sketch.

To show that $\mathfrak{s}(X)$ is coherent, note that the compact open sets of of $\mathfrak{s}(X)$ are precisely the clopen upsets of X.

To show that $\mathfrak{s}(X)$ is sober, observe that the closed sets are exactly the closed downsets, Consequently, the closed irreducible are precisely the principle downsets of points of X. □

Patch topology

If X is a spectral space, then the patch topology $\mathcal{O}^{\#}(X)$ is generated by the subbasis

$$\{U \mid U \subseteq X \text{ compact open}\} \cup \{X \setminus V \mid V \subseteq X \text{ compact open}\}$$

Nerode realized in the 1950s that the patch topology of a spectral spaces corresponds to the Stone space of its boolean envelope.

The specialization order $\leq_X$ of a topological space X is defined by $x \leq_X y$ iff $x \in \overline{\{y\}}$.

Proposition

$\mathcal{p}at(X) = (X, \mathcal{O}^{\#}(X), \leq_X)$ is a Priestley space.

Pries = Spec

Theorem (Cornish, 1975)

The categories **Pries** *and* **Spec** *are isomorphic.*

In this sense, spectral spaces and Priestley spaces are one and the same thing.

The Cornish isomorphism plays an important role in understanding the relation between the Priestley space $\mathcal{X}(L)$ of a frame L and the space $\mathcal{pt}(L)$ of points of L.

Priestley duality for frames:

Esakia duality

Frames are complete Heyting algebras

A Heyting algebra is a bounded distributive lattice H equipped with a binary operation $\to$ such that

$$a \wedge b \leq c \iff a \leq b \to c$$

for all $a, b, c \in H$.

Lemma

$D \in \mathbf{DLat}$ is a frame iff D is a complete Heyting algebra.

Proof.

Homework. □

End of Part 1