Priestley duality and its applications to pointfree topology

Sebastian David Melzer

New Mexico State University

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Structure of the course

Background
- Pointfree topology
- Stone duality and spectral spaces

Priestley duality for frames
- Priestley spaces
- Esakia duality
- Pultr-Sichler duality

Pointfree topology in the language of Priestley spaces
- Spatiality
- Sublocales
- Compactness
- Continuity and regularity
- Various dualities

Recap from Last Time

We discussed how the pt functor assigns each frame a space, though this space may not always serve as an ideal representative.

To address this, we considered the category of locales, however this lacks the point-based intuition of traditional topological spaces.

We then revisited Stone duality for bounded distributive lattices, noting how it provides a topological representation for the category of frames through special spectral spaces.

Lastly, we introduced Priestley spaces and their relationship to spectral spaces. Today, we will continue exploring Priestley spaces, focusing on the unique properties of Priestley spaces of frames.

Priestley duality for frames:

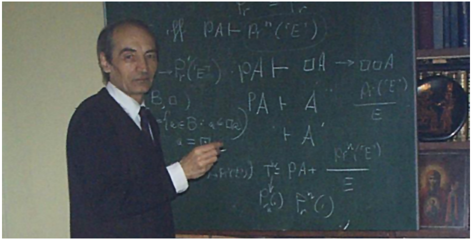
Esakia duality

Frames are complete Heyting algebras

Lemma

$D \in \mathbf{DLat}$ is a frame iff D is a complete Heyting algebra.

Duality for Heyting algebras was developed by Leo Esakia.



Since Heyting algebras are bounded distributive lattices, the focus now shifts to identifying the unique properties that Priestley spaces of Heyting algebras exhibit.

Implication in $\mathsf{ClopUp}(X)$

Let H be a Heyting algebra and X its Priestley space. Since $H \cong \mathsf{ClopUp}(X)$, the implication $\to$ exists in $\mathsf{ClopUp}(X)$.We can compute:

$$\begin{aligned} U \to V &= \bigvee\{C \mid U \cap C \subseteq V\} \\ &= \bigvee\{C \mid C \subseteq V \cup U^c\}, \end{aligned}$$

so $U \to V$ is the largest clopen upset contained in $V \cup U^c$.

To find the largest upset contained with a subset $Z \subseteq X$, we use $\downarrow(Z^c)^c$. Therefore,

$$\begin{aligned} U \to V &= (\downarrow(V \cup U^c)^c)^c \\ &= (\downarrow(U \setminus V))^c \end{aligned}$$

Only the downset operation in this expression does not inherently preserve clopenness. Thus, for $\to$ to exist, the downset of a clopen must also be clopen.

Esakia spaces

This condition brings us to the concept of Esakia spaces.

Definition

An Esakia space is a Priestley space X such that $\downarrow U$ is clopen for each clopen $U \subseteq X$.

The following is an equivalent characterization.

Lemma

Let X be a Priestley space. Then X is an Esakia space iff $\overline{U}$ is an upset for each upset $U \subseteq X$.

Esakia duality

HA – Heyting algebras and their homomorphisms

Esa – Esakia spaces and their morphisms

Theorem (Esakia, 1974)

HA *and* **Esa** *are dually equivalent.*

However, not every frame homomorphism respects the structure of a Heyting algebra, to address this, we change these to full subcategories of **DLat** and **Pries**:

$\mathbf{HA}_{\heartsuit}$ – Heyting algebras and bounded lattice homomorphisms

$\mathbf{Esa}_{\heartsuit}$ – Esakia spaces and Priestley morphisms

Theorem

$\mathbf{HA}_{\heartsuit}$ *and* $\mathbf{Esa}_{\heartsuit}$ *are dually equivalent.*

A picture

We arrive at the following situation.

$$\begin{array}{ccc} \mathbf{DLat} & \leftrightsquigarrow & \mathbf{Pries} \\ \Big| & & \Big| \\ \mathbf{HA}_{\heartsuit} & \leftrightsquigarrow & \mathbf{Esa}_{\heartsuit} \end{array}$$

Our next goal is to add **Frm** to this diagram.

For this we need to understand completeness in terms of Esakia spaces.

Arbitrary joins in Priestley spaces

Let D be a bounded distributive lattice and X its Priestley space.

Lemma

Let $S \subseteq D$. Then $\bigvee S$ exists iff $\uparrow\overline{\bigcup \varphi[S]}$ is clopen.

Recall, that $\mathfrak{s}(X) = (X, \mathsf{OpUp}(X))$ is a spectral space, where the compact opens are precisely the clopen upsets. Therefore, $\mathsf{ClopUp}(X)$ forms a basis for $\mathfrak{s}(X)$, meaning every open upset in a Priestley space can be expressed union of clopen upsets.

Theorem

D is complete iff for $\uparrow\overline{U}$ is clopen for each open upset $U \subseteq X$.

Extremally order-disconnected spaces

If D is an Heyting algebra, then X is an Esakia space. Hence, $\overline{\bigcup\varphi[S]}$ is already an upset, so $\uparrow$ is no longer required.

Corollary

$D \in \mathbf{HA}$ is complete iff $\overline{U}$ is clopen for each open upset U.

If, additionally, D is a Boolean algebra, the order becomes trivial.

Corollary

$D \in \mathbf{BA}$ is complete iff $\overline{U}$ is clopen for each open set U.

Spaces satisfying this condition are called extremally disconnected.

The order-based analogue is known as extremally order-disconnected.

Priestley duality for frames:

Pultr-Sichler duality

Priestley spaces of frames

We have collected all the ingredients to cook up the characteristics of Priestley spaces of frames.

Proposition

$D \in \mathbf{DLat}$ is a frame iff its Priestley space X_D is an extremally order-disconnected Esakia spaces.

Proof.
D is a frame iff its a complete Heyting algebra iff its an extremally order-disconnected Esakia space. □

Thus, Priestley spaces of frames are precisely the extremally order-disconnected Esakia spaces.

We will use a slightly different definition.

L-spaces

Lemma

A Priestley space X is an extremally order-disconnected Esakia space iff $\overline{U}$ is a clopen upset for each open upset U.

Definition

An L-space is a Priestley space such that $\overline{U}$ is a clopen upset for each open upset U.

Compare this to the alternative characterization of Esakia spaces.

Lemma

Let X be a Priestley space. Then X is an Esakia space iff $\overline{U}$ is an upset for each upset U.

L-morphisms

This covers the topological representation of frames. However, frame homomorphisms are more specific than bounded lattice homomorphisms, so we must restrict to a particular kind of Priestley morphism.

Lemma

Let $f : X \to X'$ be a Priestley morphism. Then f^{-1} is a frame homomorphism iff $\overline{f^{-1}(U)} = f^{-1}(\overline{U})$ for all open upsets $U \subseteq X'$.

Definition

An L-morphism is a Priestley morphism $f : X \to X'$ between L-spaces such that $f^{-1}\,\mathsf{cl}\,U = \mathsf{cl} f^{-1} U$ for all open upsets U of X'.

Pultr-Sichler duality

Let **LPries** be the category of L-spaces and L-morphisms.

Theorem (Pultr-Sichler, 1988)

Frm *and* **LPries** *are dually equivalent.*

Proof.

Consider the functors establishing Priestley duality:
$\mathcal{X} : \mathbf{DLat} \to \mathbf{Pries}$ and $\mathcal{D} : \mathbf{Pries} \to \mathbf{DLat}$.

By the previous lemmas, they restrict to **Frm** and **LPries**.

Moreover, the units are within **Frm** and **LPries**, so the dual equivalence restricts. □

We now have the following diagram:

$$
\begin{array}{ccc}
\mathbf{DLat} & \leftrightsquigarrow & \mathbf{Pries} \\
| & & | \\
\mathbf{HA}_{\heartsuit} & \leftrightsquigarrow & \mathbf{Esa}_{\heartsuit} \\
| & & | \\
\mathbf{Frm} & \leftrightsquigarrow & \mathbf{LPries}
\end{array}
$$

We have identified the category of Priestley spaces that correspond to frames.

Goal of this course: to introduce a framework in which every frame can be faithfully represented as a topological space, reconnecting frames with point-based intuition.

Let's see what the Priestley space of our example of a pointless frame looks like.

Example

Let L be the countable complete atomless boolean algebra. Then $\mathcal{pt}(L) = \varnothing$. But $\mathcal{X}(L)$ is (order-)homeomorphic to the Cantor space with discrete order.

This gives us a lot of points to work with.

Pointfree topology in the language of Priestley spaces:

Spatiality

Embedding points in the Priestley spcae

Let L be a frame and X its Priestley space.

There is an embedding $e : pt(L) \to X$. Without loss of generality, this embedding is identity. In fact, since we defined $pt(L)$ as the collection of completely prime filters and X as the collection of prime filters, it is the identity.

We want to understand how $pt(L)$ (with the right topology) is realized in the Priestley space X.

Localic points

First, we will characterize *localic points* (points of the frame) in the language of Priestley spaces.

Lemma

Let $x \in X$. Then $x \in pt(L)$ iff ${\downarrow}x$ is clopen.

Definition

We call $y \in X$ a localic point if ${\downarrow}y$ is clopen.

Spatiality

Being spatial is exactly having enough points.

This is realized as follows in the Priestley space.

Proposition

L is spatial iff $pt(L)$ is dense in X.

Definition

We call the collection of localic points of X, the localic part of X and denote it by Y.

Space of points

Thus, we know topologically which points of an L-space correspond to points of the frame.

However, the topology of the space of points is not the subspace topology inherited from the Priestley topology. It is the subspace topology inherited from the spectral topology on X.

Recall that for each Priestley space X, the spectral topology is the upper topology, i.e., $(X, \mathsf{OpUp}(X))$.

Lemma

Let $U \subseteq X$ be an open upset. Then $U \cap Y = \overline{U} \cap Y$.

Consequently, this subspace topology is exactly

$$\{U \cap Y \mid U \in \mathsf{ClopUp}(X)\}$$

i.e., we only need to consider the clopen upsets.

Sobrification

Recall that the space of points of a topological space is always sober. Therefore, for a topological space X the composition $\mathcal{pt}(\mathcal{O}(X))$ is called the sobrification of X.

We can see this also in the language of Priestley spaces. Let Z be a topological space. Then Z can be mapped into the Priestley space $X = \mathcal{X}(\mathcal{O}(Z))$ via the map $\lambda \colon Z \to \mathcal{pt}(\mathcal{O}(Z))$, i.e., map each point of Z to the completely prime filter of opens containing that point.

Then $\lambda[Z]$ is a subset of the localic part of X, and equality holds iff Z is sober.

An example

Let Z be the cofinite topology on the natural numbers.

The (completely) prime filters of Z are exactly the principle ones and the set of all cofinite subsets.

This extra (completely) prime filter is the sobrification point.

Pointfree topology in the language of Priestley spaces:
Sublocales

Sublocales and nuclei

Recall that $S \subseteq L$ a sublocale of a frame L provided S is closed under arbitrary meets and $a \to s \in S$ for all $a \in L$ and $s \in S$.

Sublocales are the pointfree analogue of subspaces of a topological space.

It is well known that sublocales are in correspondence with nuclei:

Definition

A nucleus on a frame L is a map $j : L \to L$ satisfying

1. $a \leq ja$
2. $jja \leq ja$
3. $ja \wedge jb = j(a \wedge b)$

for all $a, b \in L$.

Nuclear subsets

Let X be an L-space.

Definition

A set $N \subseteq X$ is nuclear if it is closed and $\downarrow(U \cap N)$ is clopen for each clopen $U \subseteq X$.

Theorem

Let X be the L-space of a frame L. There is a one-to-one correspondence between.

1. *Sublocales of L.*
2. *Nuclei on L.*
3. *Nuclear subsets of X.*

For $j: L \to L$ a nucleus, $N_j = \{x \in X \mid ja \in x \implies a \in x\}$ is the corresponding nuclear subset.

Nuclear subsets versus sublocales

Let L be a frame and X its L-space.

Nuclear subset of X	**Sublocale of L**
clopen upset	open
clopen downset	closed
clopen set	open $\wedge$ closed
regular closed	regular
localic point	join-irreducible

Density

Recall, a sublocale S is dense if $0 \in S$.

Lemma

A nuclear subset N corresponds to a dense sublocale iff $\max X \subseteq N$.

We call a nuclear N subset cofinal if $\max X \subseteq N$.

Isbell's Density Theorem

Theorem

Let X be an L-space. Then $\max X$ is the least cofinal nuclear subset.

Corollary (Isbell's Density Theorem)

For a frame L, the Booleanization $\mathfrak{B}(L)$ is the least dense sublocale of L.

Proof sketch.

The key is to prove that $\max X$ is the nuclear subset corresponding to $\mathfrak{B}(L)$. □

End of Part 2