Priestley duality and its applications to pointfree topology

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Structure of the course

- Background
  - Pointfree topology
  - Stone duality and spectral spaces
- Priestley duality for frames
  - Priestley spaces
  - Esakia duality
  - Pultr-Sichler duality
- Pointfree topology in the language of Priestley spaces
  - Spatiality
  - Sublocales
  - Compactness and continuity
  - Continuity and regularity
  - Various dualities

Recap from Last Time

We discussed the properties of Priestley spaces of frames.

In particular, we saw that they are exactly the extremally order-disconnected Esakia spaces.

We then introduced the localic part of an L-space and saw how it relates to spatiality and the space of points.

Finally, we briefly discussed nuclear subsets and gave an alternative proof sketch of Isbell's Density Theorem.

Pointfree topology in the language of Priestley spaces:

Compactness and continuity

The way-below relation

In a frame L, we define the way-below relation $\ll$ by

$$a \ll b \iff b \leq \bigvee S \text{ implies } a \leq \bigvee T \text{ for some finite } T \subseteq S.$$

Lemma

$a \ll b$ iff $\varphi(b) \subseteq \overline{U}$ implies $\varphi(a) \subseteq U$ for all open upsets U.

Proof.

We will only prove the ($\Rightarrow$)-direction. Suppose $a \ll b$ and $\varphi(b) \subseteq \overline{U}$ for some open upset U. Since U is an open upset, it is of the form $U = \bigcup \varphi[S]$. Thus, $\varphi(b) \subseteq \overline{\bigcup \varphi[S]}$, which means $b \leq \bigvee S$. Therefore, $a \leq \bigvee T$ for some finite $T \subseteq S$, and hence $\varphi(a) \subseteq \bigcup \varphi[T] \subseteq U$. □

Compactness

An element a is compact if $a \ll a$. A frame is compact if its top element is compact.

Lemma

Let X be the L-space of L. $a \in L$ is compact iff $\min \varphi(a) \subseteq Y$.

Proof.

($\Rightarrow$) Suppose a is compact and let $x \in \min \varphi(a)$. We need to show $\downarrow x$ is clopen, for this it is enough to show that $U = (\downarrow x)^c$ is closed.

We will show that $\overline{U} \setminus U = \varnothing$, so suppose $z \in \overline{U} \setminus U$. Then $z \in \downarrow x$, so $z \leq x$, which gives $x \in \overline{U}$. Since $\min \varphi(a) \setminus x \subseteq U$ and $x \in \overline{U}$, we have $\varphi(a) \subseteq \overline{U}$. But a is compact, so $a \ll a$, and hence by the previous lemma $\varphi(a) \subseteq U$. Consequently $x \in U$, a contradiction. Hence, U is closed.

Lemma (⋆)

Let $U \subseteq X$ be an open upset. Then $U \cap Y = \overline{U} \cap Y$.

Proof continued.

($\Leftarrow$) Suppose $\min \varphi(a) \subseteq Y$. By the previous Lemma, we need to show $\varphi(a) \subseteq \overline{U}$ implies $\varphi(a) \subseteq U$ for all open upsets U. Thus, suppose $\varphi(a) \subseteq \overline{U}$. But then

$$\varphi(a) \cap Y \subseteq \overline{U} \cap Y = U \cap Y \subseteq U.$$

Therefore, $\varphi(a) = {\uparrow} \min \varphi(a) = {\uparrow}(\varphi(a) \cap Y) \subseteq U$, as required. □

Proposition

Let L be a frame and X its Priestley space. Then L is compact iff $\min X \subseteq Y$.

Proof.

This follows directly since $\varphi(1) = X$. □

Special Closed Upsets

Porism

The following are equivalent for closed upsets $K \subseteq X$.

1. $\min K \subseteq Y$.
2. $K \subseteq \overline{U}$ implies $K \subseteq U$ for all open upsets U.

In Priestley duality, there is a correspondence between filters of the lattice and closed upsets of the Priestley space:

Theorem (Priestley)

Let L be a bounded distributive lattice and X its Priestley dual. Then $\mathsf{Filt}(D)$ is isomorphic to $\mathsf{ClUp}(X)$.

That means there is a collection of filters of L that corresponds to closed upsets satisfying the properties of the porism.

Scott upsets

A filter $F \subseteq L$ is Scott open if $\bigvee S \in F$ implies $\bigvee T \in F$ for some finite $T \subseteq S$.

Theorem

Let $F \subseteq L$ be a filter and $K \subseteq X$ its dual closed upset. Then

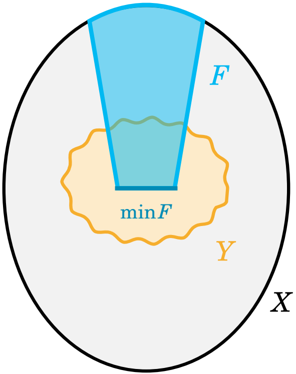
$$F \text{ is Scott open} \iff \min K \subseteq Y.$$

Definition

A Scott upset of X is a closed upset $K \subseteq X$ such that $\min K \subseteq Y$.

Thus, Scott upsets are Scott open filters in the language of Priestley spaces.

Scott upsets visually



Hofmann-Mislove

Using this correspondence, one can go on and prove the Hofmann-Mislove Theorem via Priestley duality.

Theorem (Hofmann-Mislove)

Let L be a frame and Y its space of points. Then $\mathsf{OFilt}(L)$ *is isomorphic to* $\mathsf{KSat}(Y)$.

Compare this with the theorem we saw previously:

Theorem (Priestley)

Let L be a bounded distributive lattice and X its Priestley dual. Then $\mathsf{Filt}(D)$ *is isomorphic to* $\mathsf{ClUp}(X)$.

"This theorem smells like Hofmann-Mislove."

Hofmann-Mislove and Priestley

These two theorems are not only related, they imply each other in full generality.

Priestley to Hofmann-Mislove

Let L be a frame, X its Priestley space, and Y the localic part.

$$\begin{array}{ccccc} \mathsf{Filt}(L) & \leftrightsquigarrow & \mathsf{ClUp}(X) & & \\ \Big| & & \Big| & & \\ \mathsf{OFilt}(L) & \leftrightsquigarrow & \mathsf{SUp}(X) & \longleftrightarrow & \mathsf{KSat}(Y) \end{array}$$

Hofmann-Mislove to Priestley

Let D be a bounded distributive lattice and X its Priestley space.

$$\mathsf{ClUp}(X) \longleftrightarrow \mathsf{KSat}(\mathfrak{s}(X)) \leftrightsquigarrow \mathsf{OFilt}(\mathfrak{O}(\mathfrak{s}(X))) \longleftrightarrow \mathsf{Filt}(D)$$

Pointfree topology in the language of Priestley spaces:

Continuity and regularity

The con-part of a clopen upset

A frame L is continuous if $a = \bigvee\{b \in L : b \ll a\}$. We had the following before:

Lemma

$a \ll b$ iff $\varphi(b) \subseteq \overline{U}$ implies $\varphi(a) \subseteq U$ for all open upsets U.

We can use this lemma to define the way-below relation on clopen upsets, and then use this for the following definition.

Definition

1. We define a map con: $\mathsf{ClopUp}(X) \to \mathsf{OpUp}(X)$ by
$$\operatorname{con} V = \bigcup\{W \in \mathsf{ClopUp}(X) \mid W \ll V\}.$$
2. We say con is dense if $\overline{\operatorname{con} V} = V$ for every $V \in \mathsf{ClopUp}(X)$

Theorem

L is continuous iff con is dense.

Stability

A continuous frame L is stably continuous if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$. A stably compact frame is a stably continuous frame which is compact.

Stability (and compactness) can be characterized via con.

Theorem

L is stably continuous iff con *is dense and a $\wedge$-homomorphism.*

Lemma

L is compact iff $\mathrm{con} X = X$

Theorem

L is stably compact iff con *is dense and* $(\wedge, 1)$*-homomorphism.*

The alg-part of a clopen upset

On the previous slide we saw that L is compact iff $\mathrm{con}X = X$

This is a consequence of a more general fact:

Lemma

a is compact iff $\mathrm{con}\,\varphi(a) = \varphi(a)$.

In other words, the fixpoints of con are the clopen upsets corresponding to compact elements, but those are exactly the clopen Scott upsets.

Definition

Define a map alg: $\mathsf{ClopUp}(X) \to \mathsf{OpUp}(X)$ by

$$\mathrm{alg}(U) = \bigcup\{V \in \mathsf{ClopSUp}(X) \mid V \subseteq U\}.$$

Algebraic frames

A frame is algebraic iff every element is the join of compact elements below it.

An algebraic frame is arithmetic if the binary meet of compact elements is compact

An arithmetic frame is coherent if it is compact.

Theorem

1. *L is algebraic iff* alg *is dense.*
2. *L is arithmetic iff* alg *is dense and a $\wedge$-homomorphism.*
3. *L is coherent iff* alg *is dense and a $(\wedge, 1)$-homomorphism.*

Regularity and zero-dimensionality

Regularity and zero-dimensionality can be handled in a similar way.

Definition

1. $\mathrm{reg}(U) = \bigcup\{V \in \mathsf{ClopUp}(X) \mid {\downarrow}V \subseteq U\}$
2. $\mathrm{zer}(U) = \bigcup\{V \in \mathsf{ClopBi}(X) \mid V \subseteq U\}$

where $\mathsf{ClopBi}(X)$ is the collection of clopen upsets that are also downsets.

Theorem

1. *L is regular iff* reg *is dense*
2. *L is zero-dimensional iff* zer *is dense.*

Comparison between these maps

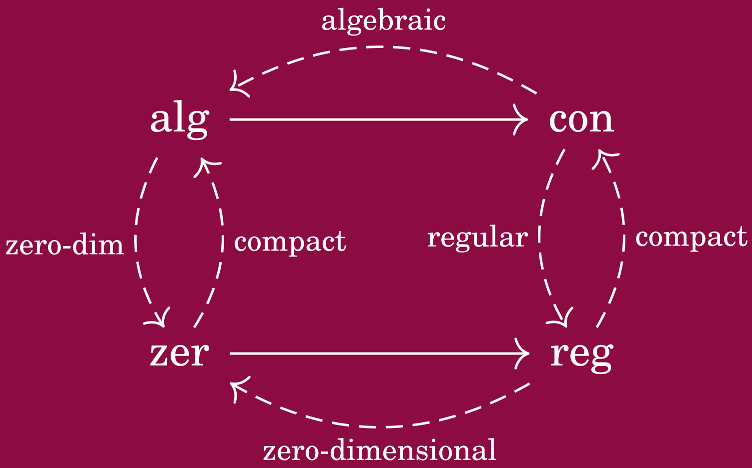
We write $\mathrm{zer} \leq \mathrm{reg}$ if $\mathrm{zer}\, U \subseteq \mathrm{reg}\, U$ for all $U \in \mathsf{ClopUp}(X)$.

We have $\mathrm{alg} \leq \mathrm{con}$. This corresponds to the fact that algebraic frames are continuous.

Similarly, $\mathrm{zer} \leq \mathrm{reg}$ mirrors the fact that zero-dimensional frames are regular.

A natural question becomes when certain other inequalities hold. For example, when $\mathrm{con} \leq \mathrm{reg}$ or vice versa?

We have a partial answer to this:


algebraic
alg
con
zero-dim
compact
regular
compact
zer
reg
zero-dimensional

Properties

There are other interesting observations. (These are not necessarily equivalent).

Frame	**L-space**
Spatial	$\overline{Y} = X$
Compact	$\min X \subseteq Y$
Regular	$Y \subseteq \min X$
Compact regular	$\min X = Y$
Stone	$\mathsf{ClopBi}(X) = \mathsf{ClopSUp}(X)$

Pointfree topology in the language of Priestley spaces:

Various dualities

LPries to Top

We have that **Frm** is dually equivalent to **LPries** and we know that there is an adjunction between **Frm** and **Top**. How do we realize this adjunction through **LPries**?

We define a functor $\mathcal{Y} : \mathbf{LPries} \to \mathbf{Top}$ by mapping an L-space X to its localic part Y, and L-morphisms $f : X \to X'$ are mapped to their restrictions to their localic parts $\mathcal{Y}(f) : Y \to Y'$.

Lemma

Let $f : X \to X'$ be an L-morphism. Then $f[Y] \subseteq Y'$.

Proof.

Let $y \in Y$ and set $U = (\downarrow f(y))^c$. Then $y \notin f^{-1}(U)$, so $\downarrow y \cap f^{-1}(U) = \varnothing$ since $f^{-1}(U)$ is an upset. But $\downarrow y$ is open, so $y \notin \overline{f^{-1}(U)} = f^{-1}(\overline{U})$ (since f is an L-morphism). Therefore, $f(y) \notin \overline{U} = \overline{(\downarrow f(y))^c}$. Consequently, $U = \overline{U}$ (if not then $f(y) \in \overline{U}$), so $\downarrow f(y)$ is open. □

Restricting $\mathscr{Y}$

Using our characterizations we can define various subcategories of **LPries**.

ConLPries – category of "continuous" L-spaces

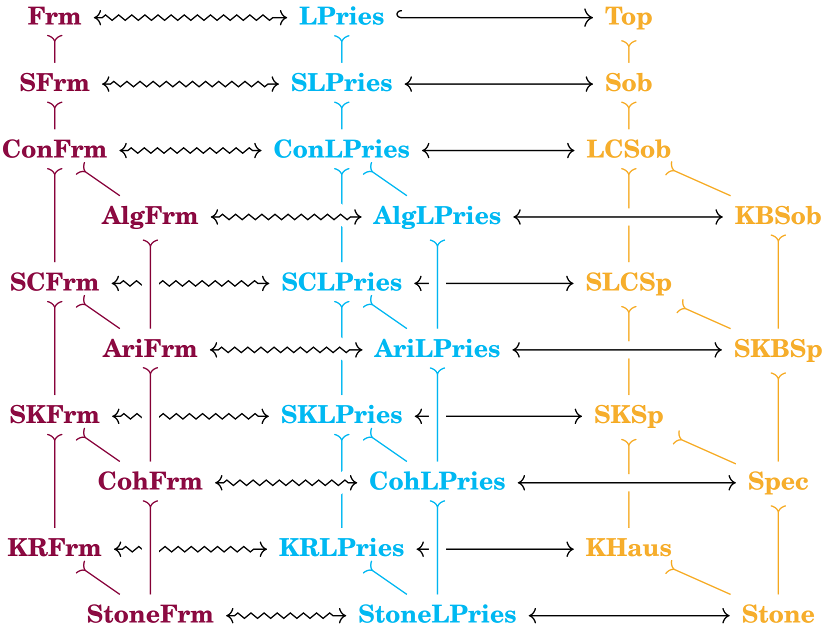
KRLPries – category of "compact regular" L-spaces

and various other categories.

$\mathscr{Y}$ restricts as expected to equivalences between these categories and the corresponding categories of topological spaces.

This yields alternative proofs of classic dualities such as:

- The Hofmann-Lawson duality between the category **ConFrm** of continuous frames and the category **LCSob** of locally compact sober spaces.
- The Isbell duality between the category **KRFrm** of compact regular frames and the category **KHaus** of compact Hausdorff spaces.


Frm
LPries
Top
SFrm
SLPries
Sob
ConFrm
ConLPries
LCSob
AlgFrm
AlgLPries
KBSob
SCFrm
SCLPries
SLCSp
AriFrm
AriLPries
SKBSp
SKFrm
SKLPries
SKSp
CohFrm
CohLPries
Spec
KRFrm
KRLPries
KHaus
StoneFrm
StoneLPries
Stone

Summary

Priestley duality is a bridge between geometric objects (ordered topological spaces) and algebraic objects (bounded distributive lattices).

In this sense, Priestley duality for frames gives a geometric perspective on pointfree topology.

However, this comes at the price of non-constructive principles: Priestley duality requires the Prime Ideal Theorem (a weaker form of the axiom of choice).

Nonetheless, this approach to pointfree topology yields several applications in pointfree topology (as well as other areas of mathematics).

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