Priestley duality for frames

Sebastian D. Melzer

New Mexico State University

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Background

Stone dualities for boolean algebras and lattices

Stone's investigation of which algebraic structures can be represented as topological spaces led to his celebrated duality.

Theorem (Stone, 1936)

The following categories are dually equivalent.

- **BA**—*boolean algebras and boolean homomorphisms.*
- **Stone**—*Stone spaces and continuous maps.*

Following this, Stone extended his duality to bounded distributive lattices.

Theorem (Stone, 1938)

The following categories are dually equivalent.

- **DLat**—*bounded distributive lattices and bounded lattice homomorphisms.*
- **Spec**—*spectral spaces and spectral maps.*

Priestley and Spectral

Priestley developed a different duality for bounded distributive lattices using compact spaces equipped with a continuous partial order.

Theorem (Priestley, 1970)

DLat *is dually equivalent to* **Pries**.

Since both **Pries** and **Spec** are dually equivalent to **DLat**, they are equivalent. In fact, they are not just equivalent but are actually isomorphic as categories.

Theorem (Cornish, 1975)

Spec *and* **Pries** *are isomorphic.*

Key idea of this talk

Since the category of frames is a subcategory of the category of bounded distributive lattices, we can refine Priestley duality and Stone duality for distributive lattice to frames.

This yields subcategories of **Pries** and **Spec** that faithfully represent the category of frames, including both spatial and non-spatial frames.

$$\begin{array}{ccccc} \mathbf{DLat} & \leftrightsquigarrow & \mathbf{Pries} & = & \mathbf{Spec} \\ | & & | & & | \\ \mathbf{Frm} & \leftrightsquigarrow & \mathbf{LPries} & = & \mathbf{LSpec} \end{array}$$

The investigation of Priestley spaces of frames was initiated by Pultr & Sichler in 1988, and spectral spaces of frames were considered by Schwartz in 2013.

In this session, we will discuss Priestley duality for frames.

Priestley duality

Priestley spaces

Definition

A Priestley space is a compact topological space X equipped with a partial order $\leq$ satisfying the Priestley separation axiom:

$$x \nleq y \implies \exists U \in \mathsf{ClopUp}(X) : x \in U \text{ and } y \notin U$$

where $\mathsf{ClopUp}(X)$ denotes the collection of clopen (=closed and open) upsets of X.

This is quite a powerful separation property. In fact, every Priestley space a Stone space.

Lemma

1. *Every closed upset / downset is an intersection of clopen upsets / downsets.*
2. *Every open upset / downset is a union of clopen upsets / downsets.*

Proof.

Let $K \in \mathsf{ClUp}(X)$. We will show that

$$K = \bigcap \{U \in \mathsf{ClopUp}(X) \mid K \subseteq U\}.$$

Clearly, the ($\subseteq$)-inclusion holds. For the converse, suppose $x \notin K$. Then $y \not\leq x$ for each $y \in K$. By PSA, there exists $U_y \in \mathsf{ClopUp}(X)$ with $y \in U_y$ and $x \notin U_y$. Then we can cover $K \subseteq \bigcup U_y$. But K is a closed subset of a compact space, so it is compact and hence

$$K \subseteq U_{y_1} \cup \cdots \cup U_{y_n}.$$

Since finite unions of clopen upsets are clopen upsets, $U = U_{y_1} \cup \cdots \cup U_{y_n} \in \mathsf{ClopUp}(X)$ with $x \notin U$. □

Proposition

Every Priestley space is a Stone space.

Proof.

Let X be a Priestley space. We need to show that X is Hausdorff and zero-dimensional.

To see that X is Hausdorff, let $x, y \in X$ be distinct. Then either $x \not\leq y$ or $y \not\leq x$. Without loss we can assume the former. By PSA, there exists $U \in \mathsf{ClopUp}(X)$ with $x \in U$ and $y \notin U$. Then U, U^c are disjoint open sets separating x and y. Thus, X is Hausdorff.

To see that X is zero-dimensional, let $U \subseteq X$ be open and $x \in U$. By the previous lemma, $\{x\} = {\uparrow}x \cap {\downarrow}x$ is an intersection of clopen sets, say $\{x\} = \bigcap \mathcal{V}$. But then $\bigcap \mathcal{V} \subseteq U$, and since X is compact, and finite intersection of clopen sets are clopen, we have $x \subseteq V \subseteq U$ for some clopen V. Therefore, X is zero-dimensional. □

Priestley space of a lattice

For each $D \in \mathbf{DLat}$, the Priestley space of D is $\mathbb{X}_D = (X_D, \tau, \subseteq)$ where X_D is the collection of prime filters, τ is generated by the subbasis

$$\{\varphi(a) \mid a \in D\} \cup \{X_D \setminus \varphi(b) \mid b \in D\},$$

where $\varphi(a) = \{x \in X_D \mid a \in x\}$ (i.e., φ is the Stone map).

Lemma

1. *$\mathbb{X}_D$ is a Priestley space.*
2. *For each $h \in \mathbf{DLat}(D, D')$, the inverse image $h^{-1} : X_{D'} \to X_D$ is a continuous order-preserving map.*

Priestley duality

Let **Pries** be the category of Priestley spaces and continuous order-preserving maps.

Theorem (Priestley, 1970)

DLat *is dually equivalent to* **Pries**.

The units of this equivalence are:

- $\varphi : D \to \mathsf{ClopUp}(X_D)$ given by $\varphi(a) = \{x \in X_d \mid a \in x\}$.
- $\epsilon : X \to X_{\mathsf{ClopUp}(X)}$ given by $\epsilon(x) = \{U \in \mathsf{ClopUp}(X) \mid x \in U\}$.

Theorem (Priestley)

$(\mathsf{Filt}(D), \subseteq)$ *is isomorphic to* $(\mathsf{ClUp}(X_D), \supseteq)$.

Proof.

Consider $\mathcal{K} : \mathsf{Filt}(D) \to \mathsf{ClUp}(X_D)$ and $\mathcal{F} : \mathsf{ClUp}(X_D) \to \mathsf{Filt}(D)$ given by

$$\mathcal{K}(F) = \bigcap\{\varphi(a) \mid a \in F\} \qquad \text{and} \qquad \mathcal{F}(K) = \{a \in D \mid K \subseteq \varphi(a)\}$$

for $F \in \mathsf{Filt}(D)$ and $K \in \mathsf{ClUp}(X_D)$. It is easy to see that they are well defined. We will show that $F = \mathcal{F}(\mathcal{K}(F))$ and $K = \mathcal{K}(\mathcal{F}(K))$.

Let $a \in F$. Then $\mathcal{K}(F) \subseteq \varphi(a)$, so $a \in \mathcal{F}(\mathcal{K}(F))$. Conversely, if $a \in \mathcal{F}(\mathcal{K}(F))$ then $\mathcal{K}(F) \subseteq \varphi(a)$, i.e., $\bigcap\{\varphi(b) \mid b \in F\} \subseteq \varphi(a)$. By compactness, $\varphi(b_1 \wedge \cdots \wedge b_n) = \varphi(b_1) \cap \cdots \cap \varphi(b_n) \subseteq \varphi(a)$. Therefore $b_1 \wedge \cdots \wedge b_n \leq a$, so $a \in F$.

Let $x \in K$. Then $x \in \varphi(a)$ for all $a \in \mathcal{F}(K)$, so $x \in \bigcap \varphi(a) = \mathcal{K}(\mathcal{F}(K))$. Conversely, if $x \notin K$ then there exists $\varphi(a)$ with $K \subseteq \varphi(a)$ and $x \notin \varphi(a)$. Hence, $a \in \mathcal{F}(K)$, so $\mathcal{K}(\mathcal{F}(K)) \subseteq \varphi(a)$. Consequently, $x \notin \mathcal{K}(\mathcal{F}(K))$. □

Priestley duality for frames

Priestley spaces of complete Heyting algebras

It is well known that frames are complete Heyting algebras.

Priestley duality was restricted to the category of Heyting algebras by Esakia in 1974.

Proposition

Let $D \in \mathbf{DLat}$ and X_D its Priestley space.

1. *D is a Heyting algebra iff* $\uparrow \mathsf{cl}\, U = \mathsf{cl}\, U$ *for each* $U \in \mathsf{OpUp}(X_D)$.
2. *D is complete iff* $\uparrow \mathsf{cl}\, U \in \mathsf{OpUp}(X_D)$ *for each* $U \in \mathsf{OpUp}(X_D)$.
3. *D is a frame iff* $\mathsf{cl}\, U \in \mathsf{OpUp}(X_D)$ *for each* $U \in \mathsf{OpUp}(X_D)$.

Definition

An L-space is a Priestley space X such that $\mathsf{cl}\, U \in \mathsf{OpUp}(X)$ for each $U \in \mathsf{OpUp}(X)$.

Frame homomorphisms

Frame homomorphisms are bounded lattice homomorphisms that additionally preserve arbitrary joins. Dually we get:

Lemma

Let $L, M \in \mathbf{Frm}$ and $h \in \mathbf{DLat}(L, M)$. Let $X_L, X_M \in \mathbf{Pries}$ and $f \in \mathbf{Pries}(X_M, X_L)$ be the dual objects. Then h is a frame homomorphism iff $f^{-1}(\mathsf{cl}\, U) = \mathsf{cl} f^{-1}(U)$ for each $U \in \mathsf{OpUp}(X_L)$.

Proof idea.

This follows from the facts that $f^{-1} \circ \varphi = \varphi \circ h$ and $\varphi(\bigvee a_i) = \mathsf{cl} \bigcup \varphi(a_i)$. □

Definition (L-morphism)

An L-morphism $f : X \to X'$ between L-spaces is a continuous order-preserving map such that $f^{-1}(\mathsf{cl}\, U) = \mathsf{cl} f^{-1}(U)$ for each $U \in \mathsf{OpUp}(X')$.

Pultr-Sichler duality

Let **LPries** be the category of L-spaces and L-morphisms.

Theorem (Pultr-Sichler, 1988)

Frm *is dually equivalent to* **LPries**.

$$\begin{array}{ccc} \mathbf{DLat} & \leftrightsquigarrow & \mathbf{Pries} \\ | & & | \\ \mathbf{Frm} & \leftrightsquigarrow & \mathbf{LPries} \end{array}$$

Completely prime filters

The space of points of a frame L is the collection of completely prime filters. Since completely prime filters are prime filters they live inside the Priestley space of L.

Lemma

$x \in X_L$ is completely prime iff $\downarrow x$ is open.

Proof.

($\Rightarrow$) We need to show that $\downarrow x$ is open. We will show that $U = (\downarrow x)^c$ is closed. Since $\downarrow x$ is closed, U is open. Therefore, $U = \bigcup \varphi(a_i)$. Then $\varphi(a_i) \subseteq (\downarrow x)^c$, which means $a_i \notin x$. Thus, $\bigvee a_i \notin x$ since x is completely prime, but $\varphi(\bigvee a_i) = \mathsf{cl} \bigcup \varphi(a_i) = \mathsf{cl}\, U$, so $x \notin \mathsf{cl}\, U$. Hence, $\mathsf{cl}\, U \subseteq (\downarrow x)^c = U$.

($\Leftarrow$) Suppose $\bigvee a_i \in x$. Then $x \in \varphi(\bigvee a_i) = \mathsf{cl} \bigcup \varphi(a_i)$. But then $\downarrow x \cap \bigcup \varphi(a_i) \neq \varnothing$, so $x \in \varphi(a_i)$, and hence $a_i \in x$. □

Localic points and spatiality

Definition

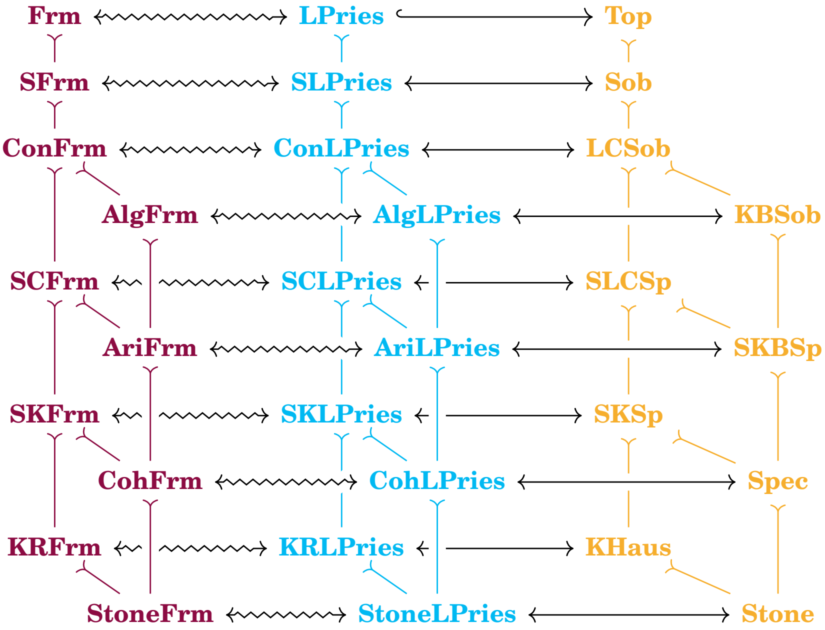
A point $y \in X$ is called localic if $\downarrow y$ is open. The collection of localic points of X is denoted by Y and called the localic part.

If X_L is the Priestley space of a frame L then the localic part Y_L can be thought of as the space of points of L.

Aside: a frame is spatial if it has enough points (= completely prime filters). In the language of Priestley:

Proposition

L is spatial iff Y_L is dense in X_L.


Frm
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Top
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ConFrm
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LCSob
AlgFrm
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SCLPries
SLCSp
AriFrm
AriLPries
SKBSp
SKFrm
SKLPries
SKSp
CohFrm
CohLPries
Spec
KRFrm
KRLPries
KHaus
StoneFrm
StoneLPries
Stone

Scott open filters

Scott upsets

A filter $F \subseteq L$ of a frame is Scott open if $\bigvee S \in F$ implies $\bigvee T \in F$ for some finite $T \subseteq S$.

Proposition

Let $F \subseteq L$ be a filter and $K \subseteq X_L$ its dual closed upset ($K = \mathcal{K}(F)$). Then F is Scott open iff $\min K \subseteq Y$.

Proof.
($\Rightarrow$) Suppose F is Scott open. We will show $\min K \setminus Y = \varnothing$, so suppose $x \in \min K \setminus Y$. Then $U = (\downarrow x)^c$ is not closed, so $x \in \mathsf{cl}\, U$. Moreover, $\min K \setminus x \subseteq (\downarrow x)^c = U \subseteq \mathsf{cl}\, U$. Hence, $\min K \subseteq \mathsf{cl}\, U$, and therefore $K = \uparrow \min K \subseteq \mathsf{cl}\, U$. But $U = \bigcup \varphi(a_i)$, so $K \subseteq \mathsf{cl} \bigcup \varphi(a_i) = \varphi(\bigvee a_i)$. Hence, $\bigvee a_i \in F$, and therefore $a_i \in F$, which gives $K \subseteq \varphi(a_i)$.

($\Leftarrow$) Omitted. □

Scott upsets

Definition

A Scott upset of X is a closed upset $K \subseteq X$ such that $\min K \subseteq Y$.

Recall we have the following theorem:

Theorem (Priestley)

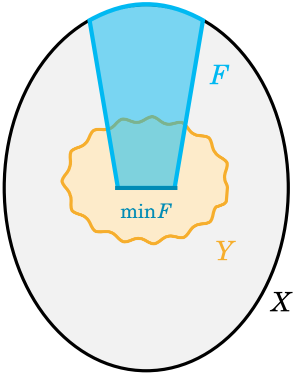
$(\mathsf{Filt}(L), \subseteq)$ *is isomorphic to* $(\mathsf{ClUp}(X_L), \supseteq)$.

By the previous proposition this restricts to Scott open filters and Scott upsets.

Corollary

$(\mathsf{SFilt}(L), \subseteq)$ *is isomorphic to* $(\mathsf{SUp}(X_L), \supseteq)$.

Scott upsets visually



Hofmann-Mislove

Theorem (Hofmann-Mislove)

$(\mathsf{SFilt}(L), \subseteq)$ *is isomorphic to* $(\mathsf{KSat}(pt(L)), \supseteq)$.

We can prove this theorem using Priestley duality by establishing a connection between Scott upsets and compact saturated sets of Y.

Theorem

$(\mathsf{SUp}(X), \supseteq)$ *is isomorphic to* $(\mathsf{KSat}(Y), \supseteq)$.

Proof sketch.

$F \in \mathsf{SUp}(X) \mapsto F \cap Y$ and $K \in \mathsf{KSat}(Y) \mapsto {\uparrow}K$. □

$$
\begin{array}{ccccc}
\mathsf{Filt}(L) & \leftrightsquigarrow & \mathsf{ClUp}(X_L) & & \\
\Big| & & \Big| & & \\
\mathsf{SFilt}(L) & \leftrightsquigarrow & \mathsf{SUp}(X_L) & \longleftrightarrow & \mathsf{KSat}(Y_L)
\end{array}
$$

Admissible filters

Admissible filters

Recall that nuclei are special maps on a frame that correspond to sublocales.

Definition

Let L be a frame. A nucleus is a map $j\colon L \to L$ satisfying

$$a \leq ja \qquad jja = ja \qquad j(a \wedge b) = ja \wedge jb$$

for all $a, b \in L$. Let $N(L)$ be the frame of nuclei on L.

For each $j \in N(L)$, there is a filter $F_j = \{a \in L \mid ja = 1\}$. We will call filters of this form admissible.

Note, each nucleus gives rise to a admissible filter, but there might be multiple nuclei with the same admissible filter. There is a one-to-one correspondence between admissible filters and fitted nuclei.

Nuclear subsets

Nuclei on L correspond to special closed subsets of X_L.

Definition

A nuclear subset $N \subseteq X$ is a closed set such that $\downarrow(N \cap U)$ is open for each open $U \subseteq X$. Let $N(X)$ be the coframe of nuclear subsets of X.

The following was proved in a slightly different context.

Theorem (Bezhanishvili & Ghilardi, 2007)

$N(L) \cong N(X_L)^{op}$.

Localic points are nuclear

Lemma

$x \in X$ is localic iff $\{x\}$ is nuclear

Proof.

(⇒) Suppose $\downarrow x$ is open. Since $\downarrow(U \cap \{x\}$ either equals $\varnothing$ or $\downarrow x$, both of which are open.

(⇐) If $\{x\}$ is nuclear then $\downarrow(X \cap \{x\}) = \downarrow x$ is open, so x is localic. □

Proposition

$\mathsf{cl}(Z \cap Y) \in N(X)$ *for every* $Z \subseteq X$.

Proof.

For $N_i \in N(X)$, we have $\mathsf{cl} \bigcup N_i \in N(X)$. Since $\mathsf{cl}(Z \cap Y) = \mathsf{cl} \bigcup_{y \in Z \cap Y} \{y\}$ we get the result from the lemma. □

Admissibly in terms of Priestley

Each nuclei gives rise to an admissible filter. We now describe this situation dually.

Lemma

Let $j \in N(L)$ and $N_j \in N(X_L)$ its corresponding nuclear subset. Then $\mathcal{K}(F_j) = {\uparrow}N_j$, i.e., the admissible filter F_j corresponds to the closed upset ${\uparrow}N_j$.

Proposition

A filter $F \subseteq L$ is admissible iff $\mathcal{K}(F) = {\uparrow}N$ for some $N \in N(X_L)$.

Scott open filters are admissible

Theorem

Scott open filters are admissible.

Proof.

If $F \subseteq L$ is a Scott open filter, then $K = \mathcal{K}(F)$ is a Scott upset, which means $\min K \subseteq Y$. However, $\mathsf{cl}(K \cap Y)$ is nuclear, and

$$K = \uparrow \min K \subseteq \uparrow \mathsf{cl}(K \cap Y) \subseteq K.$$

Hence, F is admissible. $\square$

Let L be a frame and X_L its Priestley space.

Filter $F \subseteq L$	Closed upset $K \subseteq X_L$
Prime filter	$K = {\uparrow}x$ for $x \in X_L$
Completely prime filter	$K = {\uparrow}y$ for $y \in Y_L$
Admissible filter	$K = {\uparrow}N$ for $N \in N(X_L)$
Scott open filter	$K = {\uparrow}(K \cap Y_L)$

Thanks