POINTFREE TOPOLOGY AND PRIESTLEY DUALITY

BY
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Dedication

To those who choose tolerance

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Publications

1. G. Bezhanishvili, P. Bhattacharjee, and S. D. Melzer. Maximal d-spectra via Priestley duality. *arXiv:2501.07673*, 2025. Submitted
2. G. Bezhanishvili and S. D. Melzer. Algebraic frames in Priestley duality. *Algebra Universalis*, 86(1):Paper No. 6, 25 pp., 2025
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Abstract

Pointfree topology provides an alternative approach to classical topology. A fundamental aspect of this approach is the adjunction between the categories of topological spaces and frames, which restricts to a dual equivalence between sober spaces and spatial frames, leaving non-spatial frames outside this equivalence. Priestley duality, originally developed for bounded distributive lattices, restricts naturally to frames, allowing them to be studied via their associated Priestley spaces. This perspective is particularly useful when traditional spatial representations are insufficient.

In this work, we develop a unified framework for characterizing important subcategories of frames through their Priestley spaces. By establishing equivalences between the corresponding categories of Priestley spaces and significant classes of topological spaces, we systematically derive key results in pointfree topology. In particular, we obtain a new proof of the Hofmann–Mislove Theorem and demonstrate how it leads to classic dualities, such as Hofmann–Lawson duality for continuous frames, Isbell duality for compact regular frames, and Stone dualities for coherent and Stone frames.

A key advantage of this approach is that it unifies algebraic and topological perspectives. While the adjunction between topological spaces and frames fully characterizes spatial frames, it does not extend to non-spatial ones. Priestley duality fills this gap by associating frames with dual ordered topological spaces, offering new insights into the interplay between pointfree and classical topology.

Contents

Introduction

Pointfree topology (see, e.g., [Joh82, PP12]) offers an alternative to the traditional study of topology, which typically focuses on points and their neighborhoods. Instead, it emphasizes the lattice of open sets as the fundamental object of study. This shift is motivated both constructively and categorically: reasoning about spaces without explicit reference to points avoids reliance on strong assumptions such as the Axiom of Choice while enabling an algebraic approach to topology. The key objects in this framework are *frames* (also called *locales*), which are complete lattices in which arbitrary joins distribute over finite meets.

By replacing the study of point-sets with an analysis of the lattice of open sets, pointfree topology provides a powerful algebraic perspective on topological spaces. This viewpoint has led to deep connections between topology, lattice theory, and category theory. A fundamental aspect of this approach is the relation between the categories Frm of frames and Top of topological spaces. While the classical adjunction between Top and Frm (see, e.g., [DP66]) restricts to a dual equivalence (see, e.g., [Joh82, Sec. II-1]) between spatial frames (those corresponding to the topology of a space) and sober spaces, many non-spatial frames remain outside this equivalence, highlighting the distinct algebraic landscape of pointfree topology.

Priestley duality provides a powerful tool for bridging the gap between frames and their topological counterparts by associating each frame with a Priestley space. This duality, originally introduced by Priestley [Pri70, Pri72], establishes a categorical equivalence between bounded distributive lattices and compact ordered spaces satisfying the Priestley separation axiom.

Beyond its role in duality theory, Priestley duality has been extensively applied in lattice theory and related areas (see, e.g., [GvG24]). A significant development in this area was initiated by Pultr and Sichler [PS88], who demonstrated that Priestley duality restricts to a dual equivalence between the category of frames and a subcategory of Priestley spaces, denoted LPries. The objects of LPries, called L-spaces (so named to reflect their connection to locales), are Priestley spaces satisfying additional structural constraints. This work laid the foundation for further research characterizing properties of frames in terms of their Priestley duals. Subsequent contributions [PS00, BG07, BGJ13, BGJ16, ABMZ20, ABMZ21] refined and extended these results, providing insights into various properties of frames and their dual spaces. An alternative approach, employing spectral rather than Priestley spaces, was explored in [Sch13, Sch17a, Sch17b, DST19]. Since the categories of Priestley and spectral spaces are isomorphic (see, e.g., [Cor75]) this approach provides a different yet equivalent viewpoint.

Outside of pointfree topology, the study of Priestley spaces of frames has found applications in other areas, particularly logic. One notable connection arises in the theory of nuclei, which play a fundamental role in pointfree topology as kernels of frame homomorphisms [PP12, p. 31]. Nuclei also appear in modal logic, modeling the so-called lax modality [Gol81, FM97], with applications in various branches of logic [FM95, AMdPR01, GA08, Gol11, AP16, BH16]. As demonstrated in [BH19], they provide a unified semantic hierarchy for intuitionistic logic. Importantly, the structure of the frame of nuclei on a given frame has been effectively analyzed using the language of Priestley spaces, leading to significant insights into its complexity (see, e.g., [ABMZ20, ABMZ21]).

This dissertation explores the role of Priestley duality in pointfree topology, particularly its function as an intermediary between frames and topological spaces. While several classic results in topology and domain theory have well-established proofs, a central theme of this work is demonstrating how Priestley duality provides a fresh perspective on these results, yielding streamlined proofs and new structural insights. We emphasize that Priestley duality relies on the Prime Ideal Theorem—a choice principle weaker than the full Axiom of Choice. Therefore, the results in this thesis will depend on it.

A primary focus of this study is understanding frames through their Priestley spaces. By leveraging Priestley duality, we provide new characterizations of important subcategories of frames, including spatial, continuous, and algebraic frames, as well as various other subcategories. This leads to alternative proofs of key dualities in pointfree topology, such as Hofmann–Lawson duality for continuous frames [HL78], Isbell duality for compact regular frames [Isb72], and Stone dualities for both coherent and Stone frames [Ban89, Jak13].

Beyond categorical dualities, we apply Priestley duality to revisit the Hofmann–Mislove Theorem [HM81], a fundamental result at the intersection of domain theory and pointfree topology (see, e.g., [GHK+03]). By reinterpreting this theorem through the lens of Priestley duality, we establish a framework that not only yields alternative proofs of the dualities mentioned above, but also provides new insights into the structure of continuous and algebraic frames. A central feature of our approach is the characterization of different Priestley spaces of frames via distinct *kernels*, which are maps on a Priestley space. This perspective reinforces the role of Priestley duality as a unifying tool in pointfree topology, demonstrating its broad applicability in understanding the algebraic and topological aspects of frames.

The new categories of Priestley spaces we introduce correspond not only to important classes of frames, such as continuous and algebraic frames, but also to significant categories of topological spaces, including sober spaces, locally compact spaces, compact Hausdorff spaces, and Stone spaces. This framework offers a unifying perspective on the corresponding dualities, providing new insights and facilitating further connections between pointfree topology, lattice theory, and topology.

This dissertation is structured as follows. We present a chapter-by-chapter overview, with each chapter's introduction providing a more detailed account of its contents.

In Chapter I, we lay the foundation for our study by introducing Priestley duality in the context of frames. We begin with the classical adjunction between topological spaces and frames, which restricts to a duality between sober spaces and spatial frames, but does not account for non-spatial frames. To bridge this gap, we introduce Priestley spaces, which provide an order-topological perspective on bounded distributive lattices. This duality further restricts to frames, as in Pultr–Sichler duality, establishing the key framework for interpreting frames as topological structures.

Chapter II is primarily based on [BM22] and revisits the Hofmann–Mislove Theorem through this framework. To do this, we introduce the localic part of an L-space, which corresponds to the space of points of a frame and enables a natural characterization of the Priestley spaces of spatial frames. We then characterize Scott-open filters in terms of certain closed upsets of the Priestley space, which we call Scott upsets. Scott upsets are closely related to compactness, and we use this connection to define Priestley spaces of compact frames. The relation between Scott upsets and compactness ultimately leads to a proof of the Hofmann–Mislove Theorem.

In Chapter III, which is based on [BM23], we focus on the Priestley spaces of continuous frames. There are two perspectives on these spaces: they can be viewed as L-spaces with sufficiently many Scott upsets, or as Priestley spaces equipped with a representative kernel.

In this dissertation, we adopt the second viewpoint, as it more clearly highlights connections to Priestley spaces of other classes of frames. Using this approach, we describe the Priestley spaces of stably continuous and stably compact frames. At the end of the chapter, we examine compact regular frames, which are characterized by the coincidence of two otherwise distinct kernels. The results in this chapter can be understood as a progression from Hofmann–Lawson duality to Isbell duality, providing alternative proofs of these and other dualities.

Chapter IV is based on [BM25] and follows a structure similar to that of Chapter III, but focuses on algebraic frames. In essence, it restricts the categorical framework developed for continuous frames and L-spaces to their algebraic counterparts, further illustrating the role of kernels in describing algebraicity and zero-dimensionality. We introduce categories of Priestley spaces that are equivalent to the categories of spectral and Stone spaces, offering a new perspective on Priestley duality for bounded distributive lattices and Stone duality for Boolean algebras.

Finally, Chapter V summarizes the main findings of the dissertation and presents tables and diagrams illustrating the categories and dualities developed in the previous chapters. In addition, it includes a table of notations used throughout the dissertation (see Table 4). We conclude with directions for future work.

Chapter I

Foundations of Priestley duality for frames

Topology traditionally studies spaces through points and their neighborhoods. In contrast, pointfree topology shifts this perspective by treating the lattice of open sets as the fundamental structure, rather than individual points. This abstraction facilitates a broader algebraic approach to topology, but might come at the cost of losing some intuitive geometric interpretations.

A key challenge in pointfree topology is the lack of a faithful representation of spaces for arbitrary frames. This raises the fundamental question of whether every frame can be faithfully represented as a topological space.

This chapter introduces Priestley duality for frames, which provides a bridge between pointfree and classical topology. Priestley duality allows frames to be viewed as topological spaces equipped with an additional order structure, thereby restoring the point-based intuition.

This preliminary chapter lays the foundation for the study of frames through Priestley duality. We begin in Section 1 with a review of pointfree topology, the adjunction between topological spaces and frames, and its restriction to a dual equivalence between the categories of sober spaces and spatial frames. In Section 2, we recall Priestley duality for bounded distributive lattices, mention several properties of Priestley spaces, and discuss the isomorphism between the categories of Priestley and spectral spaces. Finally, in Section 3, we examine how Priestley duality restricts to frames and list several important properties of Priestley spaces of frames.

1 Topological spaces and frames

In this section, we recall the well-known dual adjunction between the categories of topological spaces and frames, which forms the foundation of pointfree topology. We review the functors establishing this adjunction and discuss its restriction to a dual equivalence between sober spaces and spatial frames. While this section primarily introduces the fundamental adjunction, additional notions from pointfree topology will be developed in later sections as they become relevant. This approach allows for a structured development of the theory, introducing each concept in its natural context.

The results and definitions in this section are well known; for more details we refer to [Joh82, PP12].

Definition 1.1. A *frame* is a complete lattice L in which arbitrary joins distribute over binary meets, meaning that for all $S \subseteq L$ and $a \in L$, the *frame law* holds:

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}.$$

Remark 1.2. Recall (see, e.g., [Joh82, p. 8]) that a Heyting algebra is a bounded distributive lattice H equipped with a binary operation $\to$ such that $(c \wedge a) \leq b$ iff $c \leq (a \to b)$ for all $a, b, c \in H$. A lattice is a frame iff it is a complete Heyting algebra (see, e.g., [Joh82, p. 39]).

A fundamental example of a frame is the lattice of open sets $\Omega(X)$ of any topological space X. However, not every frame arises this way (see Example 1.11).

Definition 1.3. A frame is *spatial* if it is isomorphic to $\Omega(X)$ for some topological space X.

Not every space X is uniquely determined by $\Omega(X)$ (e.g., any indiscrete space has the same frame of opens). This raises the question of how topological spaces and frames relate categorically, which we examine through the functors Ω and pt.

A continuous map $f\colon X \to Y$ between topological spaces induces a bounded lattice homomorphism $\Omega(f) \coloneqq f^{-1}\colon \Omega(Y) \to \Omega(X)$. This homomorphism preserves arbitrary joins but not necessarily arbitrary meets, because intersections of open sets may fail to be open. This motivates the following definition:

Definition 1.4. A *frame homomorphism* is a bounded lattice homomorphism that preserves arbitrary joins.

The assignment defined by Ω forms a contravariant functor $\mathsf{Top} \to \mathsf{Frm}$, where Top denotes the category of topological spaces and continuous maps, and Frm denotes the category of frames and frame homomorphisms. Conversely, the functor $\mathrm{pt}\colon \mathsf{Frm} \to \mathsf{Top}$ sends a frame to its *space of points*.

Definition 1.5. Let L be a frame.

(1) A *point* of L is a completely prime filter of L, i.e., a filter F of L such that if $\bigvee S \in F$ for some $S \subseteq L$, then $S \cap F \neq \varnothing$.

(2) The *space of points* $\mathrm{pt}(L)$ of L is its points equipped with the topology consisting of the sets

$$\zeta(a) = \{x \in \mathrm{pt}(L) \mid a \in x\} \qquad \text{for } a \in L.$$

The functors Ω and pt establish the fundamental adjunction of pointfree topology:

Theorem 1.6 ([Joh82, Section II-1])**.** *The pair* (Ω, pt) *forms a dual adjunction between* Top *and* Frm*.*

To understand the relationship between L and $\mathrm{pt}(L)$, it is useful to examine when $\mathrm{pt}(\Omega(X))$ recovers the original space X. Every point x in a topological space X defines a completely prime filter

$$F_x = \{U \in \Omega(X) \mid x \in U\},$$

but not every completely prime filter arises this way.

Example 1.7. In the cofinite topology on an infinite set X, the collection of all cofinite sets forms a completely prime filter, yet it is not of the form F_x for any $x \in X$.

Definition 1.8. A space X is *sober* if every irreducible closed subset (one that cannot be written as the union of two proper closed subsets) is the closure of a unique singleton.

Sobriety ensures that X is uniquely determined by $\Omega(X)$, and that $\mathrm{pt}(\Omega(X))$ is homeomorphic to X. In particular, this means that every completely prime filter of $\Omega(X)$ is of the form F_x for some unique $x \in X$.

Remark 1.9. In a Hausdorff space, the only irreducible closed sets are singletons. Consequently, every Hausdorff space is sober (see, e.g., [Joh82, p. 43]).

By restricting the adjunction to sober spaces and spatial frames, we obtain an equivalence of categories. Let Sob denote the full subcategory of Top consisting of sober spaces, and let SFrm denote the full subcategory of Frm consisting of spatial frames.

Theorem 1.10 (see, e.g., [Joh82, p. 44])**.** Sob *is dually equivalent to* SFrm*.*

We thus arrive at Fig. 1, where $\mathsf{A} \leftrightsquigarrow \mathsf{B}$ indicates that A and B are dually equivalent, $\mathsf{A} \rightleftharpoons \mathsf{B}$ that there is a dual adjunction between A and B, and $\mathsf{A} \hookrightarrow \mathsf{B}$ that A is a full subcategory of B. The numbers next to the arrows denote where the corresponding adjunctions or equivalences are stated.

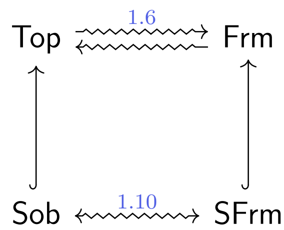

Figure 1: The fundamental adjunction between topological spaces and frames.

Theorem 1.10 demonstrates that frames faithfully generalize sober spaces. This highlights the role of completely prime filters in distinguishing spatial frames and sober spaces. However, some frames lack completely prime filters entirely, as we now illustrate.

Example 1.11 (see, e.g., [PP12, p. 19])**.** Consider a complete atomless Boolean algebra B, such as the lattice of regular open sets of the real line. Since every complete Boolean algebra is a frame, B is a frame. However, completely prime filters in B correspond to atoms, and because B is atomless, it has no completely prime filters, meaning that $\mathrm{pt}(B) = \varnothing$. This shows that B is as far from being spatial as possible.

This example indicates that frames can behave significantly differently from classical topological spaces. Nonetheless, frames remain amenable to topological methods, particularly through Priestley duality, which provides a powerful framework for interpreting all frames, even those with no points at all.

2 Priestley duality for distributive lattices

The adjunction between topological spaces and frames provides a foundation for pointfree topology, but it does not fully capture the structure of frames from a topological perspective.

A different duality is needed to bridge the gap. In this section, we introduce Priestley duality, which establishes a connection between bounded distributive lattices and certain ordered topological spaces known as *Priestley spaces*.

The definition and results in this section are well known; for more details we refer to, e.g., [Pri84, GvG24]. Recall that an *upset* of a partially ordered set $(X, \leq)$ is a subset Z such that

$$Z = {\uparrow} Z \coloneqq \{x \in X \mid z \leq x \text{ for some } z \in Z\}.$$

Downsets and ${\downarrow} Z$ are defined analogously.

Definition 2.1. A *Priestley space* is a compact topological space X equipped with a partial order $\leq$ satisfying the *Priestley separation axiom*:

$$x \not\leq y \text{ implies that there exists a clopen upset } U \text{ such that } x \in U \text{ and } y \notin U \tag{I.1}$$

for all $x, y \in X$.

To each bounded distributive lattice, we assign the Priestley space consisting of its prime filters:

Definition 2.2. Let L be a bounded distributive lattice. The *Priestley space* $\mathcal{X}(L)$ *of* L is the collection of its prime filters equipped with:

- the topology generated by the subbasis

  $$\{\varphi(a) \mid a \in L\} \cup \{\mathcal{X}(L) \setminus \varphi(b) \mid b \in L\}$$

  where $\varphi(a) \coloneqq \{x \in \mathcal{X}(L) \mid a \in x\}$ for each $a \in L$, and
- the partial order given by the subset inclusion $\subseteq$.

This construction ensures that $\mathcal{X}(L)$ is a Priestley space for each bounded distributive lattice L. A bounded lattice homomorphism $h\colon L \to M$ induces a continuous order-preserving map $h^{-1}\colon \mathcal{X}(M) \to \mathcal{X}(L)$. This motivates the following definition:

Definition 2.3. A *Priestley morphism* is a continuous order-preserving map $f\colon X \to Y$ between Priestley spaces, meaning that $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$.

The category Pries consists of Priestley spaces and Priestley morphisms, while the category DLat consists of bounded distributive lattices and bounded lattice homomorphisms. Priestley duality is obtained via the following functors:

- The functor $\mathcal{X}\colon \mathsf{DLat} \to \mathsf{Pries}$ maps each bounded distributive lattice to its Priestley space and sends a bounded lattice homomorphism $h\colon L \to M$ to the Priestley morphism $h^{-1}\colon \mathcal{X}(M) \to \mathcal{X}(L)$.
- The functor $\mathcal{C}lopUp\colon \mathsf{Pries} \to \mathsf{DLat}$ assigns to each Priestley space X its lattice of clopen upsets $\mathrm{ClopUp}(X)$ and sends a Priestley morphism $f\colon X \to Y$ to the bounded lattice homomorphism $f^{-1}\colon \mathrm{ClopUp}(Y) \to \mathrm{ClopUp}(X)$.

Theorem 2.4 (Priestley duality; [Pri70, Pri72])**.** Pries *is dually equivalent to* DLat*.*

Remark 2.5. One of the units of this dual equivalence is $\varphi\colon L \to \mathrm{ClopUp}(\mathcal{X}(L))$. This ensures that for each bounded lattice homomorphism $h\colon L \to M$ the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & \mathrm{ClopUp}(\mathcal{X}(L)) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{C}lopUp\,\mathcal{X}(h)} \\ M & \xrightarrow[\varphi]{} & \mathrm{ClopUp}(\mathcal{X}(M)) \end{array}$$

In particular, if $f = \mathcal{X}(h)$ then $f^{-1}(\varphi(a)) = \varphi(h(a))$ for each $a \in L$. This will be used in Chapter III.

The Priestley separation axiom is a strong separation property, implying that every Priestley space is zero-dimensional and Hausdorff, making it a Stone space.

Definition 2.6. A *Stone space* is a zero-dimensional compact Hausdorff space.

Let BA be the category of Boolean algebras with their homomorphisms and let Stone be the full subcategory of Top consisting of Stone spaces. *Stone duality*, which establishes an equivalence between BA and Stone, arises as a special case of Priestley duality. Specifically, if we consider a Boolean algebra B as a distributive lattice where every element is complemented, its Priestley space $\mathcal{X}(B)$ has a trivial order, reducing it to a Stone space. Thus, Priestley duality generalizes Stone duality by incorporating order structure.

Theorem 2.7 (Stone duality; [Sto36])**.** Stone *is dually equivalent to* BA*.*

We can view Stone as the full subcategory of Pries of Priestley spaces where the order is trivial. This yields the relationships shown in Fig. 2.

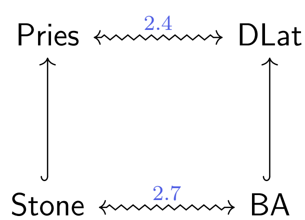

Figure 2: Priestley duality as an extension of Stone duality.

We now briefly list some known properties of Priestley spaces that will be used throughout this thesis. In a Stone space, the collection of clopen sets forms a basis (as it is zero-dimensional). A similar role is played by the clopen upsets and downsets of a Priestley space. This is made precise by the following result. We denote the clopen downsets of X by $\mathrm{ClopDn}(X)$.

Lemma 2.8 (see, e.g., [PS88, Prop. 1.3]). *Let X be a Priestley space.*

(1) $\mathrm{ClopUp}(X) \cup \mathrm{ClopDn}(X)$ *forms a subbasis for X.*

(2) *Every open upset (resp. downset) is a union of clopen upsets (resp. downsets).*

(3) *Every closed upset (resp. downset) is an intersection of clopen upsets (resp. downsets).*

Closed subsets of a Priestley space interact well with order, as seen below. For a poset Z, we write $\min Z$ and $\max Z$ for the minimal and maximal elements of Z, respectively.

Lemma 2.9 (see, e.g., [Pri84, Prop. 2.6]). *Let X be a Priestley space and let $F \subseteq X$ be closed.*

(1) *Both $\uparrow F$ and $\downarrow F$ are closed.*

(2) *Both $\min F$ and $\max F$ are nonempty. In fact, for every $x \in F$ there exist $y \in \min F$ and $z \in \max F$ such that $y \leq x \leq z$. Consequently, if F is a closed upset, then $F = \uparrow \min F$, and if F is a closed downset, then $F = \downarrow \max F$.*

By Priestley duality, points of a Priestley space can be identified with prime filters of its lattice of clopen upsets. This leads to the following result, which is used in Section 4 to extend maps between subsets of Priestley spaces to the entire space.

Proposition 2.10 (see, e.g., [GvG24, Lem. 3.25]). *Let X be a Priestley space. If $\mathscr{P}$ is a prime filter of $\mathrm{ClopUp}(X)$, then $\bigcap \mathscr{P} = \uparrow x$ for a unique $x \in X$.*

We now highlight another important consequence of Priestley duality: the correspondence between filters of a lattice and closed upsets of its Priestley space. This will play an important role in Section 6.

Let $\mathrm{Filt}(L)$ denote the collection of filters of a lattice L and $\mathrm{ClUp}(X)$ the collection of closed upsets of a Priestley space X. We view both as posets under inclusion.

Theorem 2.11 (see, e.g., [Pri84])**.** *Let L be a bounded distributive lattice and X its Priestley space.* $\mathrm{Filt}(L)$ *is dually isomorphic to* $\mathrm{ClUp}(X)$*. This isomorphism is established by the maps:*

$$F \mapsto K_F := \bigcap\{\varphi(a) \mid a \in F\} \qquad \textit{for } F \in \mathrm{Filt}(L),$$

$$K \mapsto \{a \in L \mid K \subseteq \varphi(a)\} \qquad \textit{for } K \in \mathrm{ClUp}(X).$$

We conclude this section by recalling the well-known connection between Priestley spaces and *spectral spaces*. In addition to Priestley duality, there exists another duality for bounded distributive lattices, originally developed by Stone using spectral spaces.

Definition 2.12 (see, e.g., [Hoc69, p. 43])**.** A *spectral space* is a compact sober space satisfying the following properties:

(1) It is *compactly based*, meaning that it has a basis of compact open sets.

(2) The binary intersection of two compact open sets is compact.

Remark 2.13. The term compactly based follows the terminology used in [Ern09].

Definition 2.14 (see, e.g., [Joh82, p. 64])**.** A continuous map $f\colon X \to Y$ is *coherent* if $f^{-1}(U)$ is compact for each compact open $U \subseteq Y$.

Let Spec be the category of spectral spaces and coherent maps.

Remark 2.15. Continuous maps between Stone spaces are coherent since the compact open sets of a Stone space are exactly the clopen sets. Thus, Stone is a full subcategory of Spec (see, e.g., [Joh82, p. 71]).

Theorem 2.16 ([Sto38])**.** DLat *is dually equivalent to* Spec*.*

As a consequence of Theorems 2.4 and 2.16, Spec is equivalent to Pries. In fact, these categories are isomorphic:

Theorem 2.17 ([Cor75])**.** *Pries and Spec are isomorphic.*

Remark 2.18. We briefly outline the isomorphism of Theorem 2.17. The *specialization preorder* $\leq$ on a topological space X is defined by $x \leq y$ iff $x \in \mathrm{cl}\{y\}$, where cl is the topological closure. The preorder is a partial order iff X is T_0. For spectral spaces, the *patch topology* is generated by compact open sets and their complements.

A spectral space is mapped to a Priestley space by equipping its carrier with the patch topology and the specialization order. Conversely, a Priestley space is mapped to a spectral space by equipping its carrier with the *upper topology*, which consists of the open upsets. Since these topologies are used throughout this thesis, we introduce notation: for a Priestley space X, we denote its topology (i.e., the patch topology of the corresponding spectral space) by π or π_X, and its upper topology (i.e., the topology of the corresponding spectral space) by τ or τ_X.

3 Restriction of Priestley duality to frames: Pultr–Sichler duality

Priestley duality provides a complete characterization of bounded distributive lattices in terms of ordered topological spaces. As frames form a subclass of bounded distributive lattices, they naturally fall within this framework. This section characterizes the Priestley spaces corresponding to frames.

For a Priestley space X, let $\mathrm{OpUp}(X)$ denote the collection of open upsets of X.

Definition 3.1 (L-spaces)**.**

(1) A *localic space*, or simply an L-*space*, is a Priestley space X such that

$$U \in \mathrm{OpUp}(X) \text{ implies } \mathrm{cl}\, U \in \mathrm{OpUp}(X). \tag{I.2}$$

(2) An L-*morphism* is a Priestley morphism $f\colon X \to Y$ between L-spaces such that $f^{-1}(\mathrm{cl}\, U) = \mathrm{cl}\, f^{-1}(U)$ for every $U \in \mathrm{OpUp}(Y)$.

(3) Let LPries be the category of L-spaces and L-morphisms.

The identity map and compositions of L-morphisms are L-morphisms, ensuring that LPries forms a category.

Priestley duality restricts to the categories LPries and Frm.

Theorem 3.2 (Pultr–Sichler duality; [PS88])**.** *LPries is dually equivalent to Frm.*

Remark 3.3. By Theorem 2.17, there exists a category of spectral spaces isomorphic to LPries. This category was investigated in [Sch13, Sch17a, Sch17b] (see also [DST19]).

We will frequently rely on the following conditions satisfied by L-spaces:

Lemma 3.4 (see, e.g., [PS88, Sec. 2] and [Esa19, Thm. 3.12])**.** *Suppose X is an* L*-space.*

(1) *If $F \subseteq X$ is clopen, then ${\downarrow}F$ is clopen.*

(2) *If $U \subseteq X$ is open, then ${\downarrow}U$ is open.*

(3) *If $U \subseteq X$ is an upset, then $\mathrm{cl}\, U$ is an upset.*

(4) ${\uparrow}\mathrm{cl}\, F = \mathrm{cl}\,{\uparrow}F$ *for each $F \subseteq X$.*

Remark 3.5. For every Priestley space, the conditions in Lemma 3.4 are equivalent. Priestley spaces satisfying any of these conditions are known as *Esakia spaces* (see, e.g., [Esa19]).

Esakia spaces are the duals of Heyting algebras [Esa74]. That L-spaces satisfy these conditions reflects the fact that frames are complete Heyting algebras (see Remark 1.2). Moreover, Esakia spaces satisfying (I.2) are precisely the dual spaces of complete Heyting algebras.

Since the clopen upsets of an L-space form a frame, all joins exist. Throughout this thesis, we will use the following lemma to compute joins:

Lemma 3.6 (see, e.g., [BB08, Lem 2.3])**.** *Let L be a frame, X its Priestley space, and $S \subseteq L$. Then*

$$\varphi\left(\bigvee S\right) = \operatorname{cl}\bigcup \varphi[S]. \tag{I.3}$$

In the remainder of this thesis, we will utilize Pultr–Sichler duality to study frames from the perspective of Priestley spaces. In particular, we will restrict Pultr–Sichler duality to several important categories of frames and characterize their Priestley spaces.

Chapter II

Hofmann–Mislove through the lenses of Priestley

The Hofmann–Mislove Theorem [HM81] establishes a fundamental connection between topological structures and domain theory (see [GHK+03, pp. 144–150]). Given a sober space X and its frame of opens $L = \Omega(X)$, the theorem states that the poset $\mathrm{OFilt}(L)$ of Scott-open filters of L is isomorphic to the poset $\mathrm{KSat}(X)$ of compact saturated subsets of X. This, in particular, implies that $\mathrm{KSat}(X)$ is a domain for a locally compact sober space X (since $\mathrm{OFilt}(L)$ is a domain for a locally compact space; see [GHK+03, p. 145]). Since its original proof in 1981, the theorem has been revisited in various settings, with Keimel and Paseka's proof [KP94] considered one of the most direct and widely used.

A structurally similar result exists in Priestley duality (see Theorem 2.11), where the poset of filters of a bounded distributive lattice corresponds to the poset of closed upsets of the Priestley space. A close look at the two proofs reveals striking similarities. Indeed, it was pointed out in [BBGK10, Rem. 6.4] that the two results are equivalent in the setting of spectral spaces.

In this chapter we demonstrate that Priestley duality for frames provides a natural framework for proving the Hofmann–Mislove Theorem. By interpreting Scott-open filters in terms of closed upsets of the Priestley space, we establish a more general version of the theorem. Specifically, we will prove that $\mathrm{OFilt}(L)$ is isomorphic to $\mathrm{KSat}(\mathrm{pt}(L))$ for an arbitrary frame L by showing that $\mathrm{OFilt}(L)$ corresponds to the poset of Scott upsets of the Priestley space of L. The Hofmann–Mislove Theorem is an immediate consequence.

This chapter is organized as follows. Section 4 develops the notion of the localic part of an L-space, providing a Priestley perspective on spatial frames and revisiting the equivalence of Theorem 1.10. Section 5 builds on this by analyzing compact frames, compact elements, and the way-below relation in terms of Priestley spaces, leading naturally to the concept of Scott upsets. These upsets turn out to characterize Scott-open filters in the language of Priestley spaces, and this correspondence is used in Section 6 to give a Priestley analogue of the Hofmann–Mislove Theorem, ultimately yielding a new proof of the theorem.

4 Priestley spaces of spatial frames: the localic part

This section develops the Priestley perspective on spatial frames, offering an alternative formulation of the well-known duality between Sob and SFrm (see Theorem 1.10). By shifting the focus to Priestley spaces, we recast spatiality as a purely topological property: the density of a distinguished subset.

Let L be a frame and X its Priestley space. Since completely prime filters form a distinguished subset of prime filters, the space of points $\mathrm{pt}(L)$ embeds naturally into the Priestley space X of L. The following provides a purely order-topological characterization for recognizing these points.

Lemma 4.1 ([PS00, Prop. 2.9]; see also [BGJ16, Lem. 5.1])**.** *Let L be a frame and X the Priestley space of L. Then $y \in X$ is a completely prime filter iff ${\downarrow}y$ is clopen.*

This leads to a key definition: the localic points of an L-space, which correspond to completely prime filters in the dual frame. These points and their collection will play a central role in this thesis.

Definition 4.2. Let X be an L-space.

(1) A *localic point* of X is a point $y \in X$ such that $\downarrow y$ is clopen.

(2) The *localic part* $\operatorname{loc} X$ of X is its collection of localic points.

Remark 4.3. Various terminologies exist for the elements of $\operatorname{loc} X$. They are called L-*points* in [PS00] and *nuclear points* in [ABMZ20]. We adopt the term localic points, following [Sch13], as it naturally reflects their role as the points of the corresponding locale.

The localic part of an L-space is crucial for understanding spatiality through Priestley duality. It provides a way to interpret spatiality of a frame in terms of a density condition on its Priestley dual, which will be formalized in the next theorem. The equivalence (1)⇔(2) is proved in [ABMZ20, Thm. 5.5] (see also [PS00, Par. 2.11]).

Theorem 4.4. *For a frame L and its Priestley space X, the following are equivalent:*

(1) *L is spatial.*

(2) *$\operatorname{loc} X$ is dense in X.*

(3) *$U \cap \operatorname{loc} X$ is dense in U for each $U \in \operatorname{ClopUp}(X)$.*

Proof. (1)⇔(2) is known (see above).

(2)⇔(3) If $\operatorname{loc} X$ is dense in X, then $U \cap \operatorname{loc} X$ is dense in U for each open subset U of X. The other implication follows immediately because $X \in \operatorname{ClopUp}(X)$. □

Theorem 4.4 formalizes the idea that having "enough points" in the frame corresponds to having "enough localic points" in the Priestley dual. The inclusion of the third condition of Theorem 4.4 will become more illustrative as we explore different density conditions that characterize other classes of frames. Many important frame properties can be understood through variations of this condition by identifying various "kernels" of clopen upsets that are required to be dense in the clopen upsets.

Definition 4.5.

(1) An L-space X is L-*spatial* if $\operatorname{loc} X$ is dense in X.

(2) Let $\mathsf{SLPries}$ be the full subcategory of LPries consisting of (L-)spatial L-spaces.

Convention 4.6. We refer to an L-space that is L-spatial as a *spatial* L-*space*, dropping the prefix "L-" when "L-spatial" modifies "L-space." This avoids unnecessary linguistic complexity, which becomes particularly relevant as additional properties of frames and their dual description in terms of L-spaces are introduced. For instance, while L-*compact* L-*regular* L-*space* is precise, it is cumbersome. However, as L-spaces are compact and regular as topological spaces, referring to an L-space as simply compact or regular may lead to confusion. To maintain clarity, we adopt the convention that the prefix "L-" may be omitted from property names only when immediately followed by "L-space."

The results above allow us to restrict Pultr–Sichler duality to spatial frames, yielding an equivalence between SFrm and $\mathsf{SLPries}$. This confirms that spatiality at the frame level corresponds precisely to L-spatiality at the Priestley space level.

Theorem 4.7. $\mathsf{SLPries}$ *is dually equivalent to* SFrm.

Proof. The result follows by restricting Theorem 3.2 to spatial frames using Theorem 4.4. □

We now define a functor from LPries to Top. To do so, we equip the localic part of an L-space with a topology and ensure that this assignment is functorial.

Definition 4.8 (The topology of the localic part)**.** Let X be an L-space. We view $\operatorname{loc} X$ as a topological space, where $U \subseteq \operatorname{loc} X$ is open iff $U = V \cap \operatorname{loc} X$ for some $V \in \operatorname{ClopUp}(X)$.

The next result confirms that this topology is precisely the topology of the space of points of the associated frame.

Lemma 4.9. *Let L be a frame and X its Priestley space.*

(1) ([ABMZ20, Lem. 5.3(1)]) $\zeta(a) = \varphi(a) \cap \operatorname{loc} X$ *for each* $a \in L$*.*

(2) ([ABMZ20, Prop. 5.4]) $\operatorname{loc} X$ *is homeomorphic to* $\operatorname{pt}(L)$*.*

Remark 4.10. Let X be a Priestley space.

(1) The topology on $\operatorname{loc} X$ is in fact the subspace topology induced by the upper topology τ (see Remark 2.18). The reason for this is that each $U \in \operatorname{OpUp}(X)$ is a union of clopen upsets (see Lemma 2.8(2)) and that $\operatorname{cl} U \cap \operatorname{loc} X = U \cap \operatorname{loc} X$ (as will be shown in Lemma 4.16(1)).

(2) If X is the Priestley space of a spatial frame L, then every $U \in \operatorname{ClopUp}(X)$ satisfies $\operatorname{cl}(U \cap \operatorname{loc} X) = U$ by Theorem 4.4(3). In particular, for each $a \in L$, Lemma 4.9(1) ensures that $\operatorname{cl} \zeta(a) = \operatorname{cl}(\varphi(a) \cap \operatorname{loc} X) = \varphi(a)$. Thus, the map $\varphi(a) \mapsto \varphi(a) \cap \operatorname{loc} X$ is an isomorphism from the lattice $\operatorname{ClopUp}(X)$ to the lattice $\Omega(\operatorname{loc} X)$.

The next result establishes that L-morphisms between L-spaces restrict to continuous maps on their localic parts.

Lemma 4.11. *Let $f \colon X \to Y$ be an* L*-morphism between* L*-spaces.*

(1) $f(\operatorname{loc} X) \subseteq \operatorname{loc} Y$*.*

(2) *The restriction $f \colon \operatorname{loc} X \to \operatorname{loc} Y$ is a well-defined continuous map.*

Proof. (1) Let $x \in \operatorname{loc} X$ and set $U := X \setminus {\downarrow} f(x)$. Since $x \notin f^{-1}(U)$ and $f^{-1}(U)$ is an upset, it follows that ${\downarrow} x \cap f^{-1}(U) = \varnothing$. Because $x \in \operatorname{loc} X$, we have that ${\downarrow} x$ is open, so

$$x \notin \operatorname{cl} f^{-1}(U) = f^{-1}(\operatorname{cl} U) \qquad \text{(see Definition 3.1(2))}.$$

Thus, $f(x) \notin \operatorname{cl} U$, and hence $f(x) \in \operatorname{int} {\downarrow} f(x)$, where int is the topological interior. It follows from Lemma 3.4(2) that $\operatorname{int} {\downarrow} f(x)$ is a downset. Consequently, since $f(x) \in \operatorname{int} {\downarrow} f(x)$, we obtain ${\downarrow} f(x) = \operatorname{int} {\downarrow} f(x)$, hence ${\downarrow} f(x)$ is open. Therefore, $f(x) \in \operatorname{loc} Y$.

(2) That the restriction of f is well defined follows from (1). For continuity, it suffices to show that $f^{-1}(U \cap \operatorname{loc} Y) \cap \operatorname{loc} X$ is open in $\operatorname{loc} X$ for every $U \in \operatorname{ClopUp}(Y)$. By (1),

$$f^{-1}(U \cap \operatorname{loc} Y) \cap \operatorname{loc} X = f^{-1}(U) \cap \operatorname{loc} X.$$

Since f is a Priestley morphism, $f^{-1}(U) \in \operatorname{ClopUp}(X)$, so $f^{-1}(U) \cap \operatorname{loc} X$ is open in $\operatorname{loc} X$. □

We define a functor $\mathscr{L}oc\colon \mathsf{LPries} \to \mathsf{Top}$ by assigning to each L-space X its localic part $\operatorname{loc} X$ and to each L-morphism $f\colon X \to Y$ its restriction $f\colon \operatorname{loc} X \to \operatorname{loc} Y$. Functoriality follows from the fact that restricting an L-morphism automatically preserves identity maps and composition.

The adjunction between Top to Frm can be formulated purely in terms of Priestley duality by defining a suitable functor from Top to Pries. Rather than introducing another functor, we establish an equivalence between $\mathsf{SLPries}$ and Sob by proving that $\mathscr{L}oc\colon \mathsf{SLPries} \to \mathsf{Sob}$ is essentially surjective, full, and faithful. We first show that the localic part of each L-space is sober.

The sobriety of the localic part follows directly from Lemma 4.9(2) as it is homeomorphic to the space of points of the associated frame. We provide an alternative proof that does not rely on this homeomorphism. It requires the following two lemmas.

Every closed subset of the localic part can be expressed as the intersection of a clopen downset with $\operatorname{loc} X$. Lemma 4.12 identifies a unique clopen downset corresponding to each closed set, providing a canonical representation.

Lemma 4.12. *Let X be an* L*-space. If $F \subseteq \operatorname{loc} X$ is closed then $\downarrow \operatorname{cl} F \in \operatorname{ClopDn}(X)$ and $F = \downarrow \operatorname{cl} F \cap \operatorname{loc} X$.*

Proof. This follows from [ABMZ20, Lem. 4.8], but for completeness we provide a proof. It suffices to show that $U := X \setminus \downarrow \operatorname{cl} F$ is closed since $\downarrow \operatorname{cl} F$ is a closed downset by Lemma 2.9(1).

Since U is an open upset, $\operatorname{cl} U \in \operatorname{ClopUp}(X)$ by (I.2). Suppose for contradiction that $F \cap \operatorname{cl} U \neq \varnothing$, meaning that there exists $y \in F$ such that $y \in \operatorname{cl} U$. Since $y \in F \subseteq \operatorname{loc} X$, the set $\downarrow y$ is open. This implies that $\downarrow y \cap U \neq \varnothing$, meaning that some point in $\downarrow y$ also belongs to U. Since U is an upset, it follows that $y \in U = X \setminus \downarrow \operatorname{cl} F$, so $y \notin \downarrow \operatorname{cl} F$. However, this contradicts $y \in F$ as $F \subseteq \downarrow \operatorname{cl} F$. Thus, we must have $F \cap \operatorname{cl} U = \varnothing$. Since $\operatorname{cl} U$ is an upset, it follows that $\downarrow F \cap \operatorname{cl} U = \varnothing$, so $\operatorname{cl} U = U$, proving that U is closed. □

Lemma 4.13. *If $V \in \operatorname{ClopDn}(X)$ is join-prime, then $V = \downarrow y$ for a unique $y \in \operatorname{loc} X$.*

Proof. This is proved dually to [BB08, Thm. 2.7(1)], where a similar statement is proved for clopen upsets. We provide the proof for completeness.

Suppose V is join-prime. By Lemma 2.9(2), we have $\downarrow \max V = V$, so it is enough to show that $\max V$ is a singleton, as this would then imply that $\downarrow y = V$ and thus that $y \in \operatorname{loc} X$.

Let $x, y \in \max V$ with $x \neq y$. It follows from the Priestley separation axiom (see (I.1)) that there exist $U_x, U_y \in \operatorname{ClopDn}(X)$ such that $x \in U_x \not\ni y$ and $y \in U_y \not\ni x$. Since $V = \downarrow \max V$, we obtain

$$V \subseteq U_x \cup \bigcup \{U_{y'} \mid y' \in \max U \text{ and } y' \neq x\}.$$

By compactness, there exist $U_{y_1}, \ldots, U_{y_n}$ such that $V \subseteq U_x \cup U_{y_1} \cup \cdots \cup U_{y_n}$. Since finite unions of clopen downsets are clopen downsets, we define $U \coloneqq U_{y_1} \cup \cdots \cup U_{y_n} \in \mathrm{ClopDn}(X)$. Thus, $V \subseteq U_x \cup U$, but since V is join-prime, it follows that either $V \subseteq U_x$ or $V \subseteq U$. However, this contradicts that $x \notin U$ and $y \notin U_x$. Therefore, $\max V$ must be a singleton, as required. □

Proposition 4.14. *Let X be an* L*-space. Then $\operatorname{loc} X$ is a sober space.*

Proof. Let $F \subseteq \operatorname{loc} X$ be an irreducible closed subset. Since open subsets of $\operatorname{loc} X$ are of the form $U \cap \operatorname{loc} X$ for some $U \in \mathrm{ClopUp}(X)$, it follows that closed subsets of $\operatorname{loc} X$ are of the form $V \cap \operatorname{loc} X$ for some $V \in \mathrm{ClopDn}(X)$. Therefore, there exists $V \in \mathrm{ClopDn}(X)$ such that $F = V \cap \operatorname{loc} X$. By Lemma 4.12, ${\downarrow} \operatorname{cl} F$ is clopen, so we may take $V \coloneqq {\downarrow} \operatorname{cl} F$. We now show that V is a join-prime element of $\mathrm{ClopDn}(X)$.

Suppose $V \subseteq D \cup E$ for some $D, E \in \mathrm{ClopDn}(X)$. Then $F \subseteq (D \cap \operatorname{loc} X) \cup (E \cap \operatorname{loc} X)$. Since F is irreducible, it follows that either $F \subseteq D \cap \operatorname{loc} X \subseteq D$ or $F \subseteq E \cap \operatorname{loc} X \subseteq E$. Consequently, $V \subseteq D$ or $V \subseteq E$, so V is join-prime.

By Lemma 4.13, we conclude that $V = {\downarrow} y$ for a unique $y \in \operatorname{loc} X$. Thus, $F = {\downarrow} y \cap \operatorname{loc} X$. Since ${\downarrow} y \cap \operatorname{loc} X$ is the least closed subset of $\operatorname{loc} X$ containing y, it follows that F is the closure of y in $\operatorname{loc} X$. Therefore, $\operatorname{loc} X$ is sober. □

Proposition 4.14 ensures that $\mathscr{L}oc\colon \mathsf{SLPries} \to \mathsf{Sob}$ is well defined.

Theorem 4.15. *$\mathscr{L}oc\colon \mathsf{SLPries} \to \mathsf{Sob}$ is essentially surjective.*

Proof. Suppose Y is a sober space. Then Y is homeomorphic to $\mathrm{pt}(\Omega(Y))$ (see Theorem 1.10). Let X be the Priestley space of $\Omega(Y)$. By Lemma 4.9(2), we have $\mathrm{pt}(\Omega(Y))$ is homeomorphic to $\operatorname{loc} X$, establishing the result. □

The following lemma describes how the closure in an L-space interacts with its localic part. In general, closure does not preserve finite meets, making Lemma 4.16(3) somewhat noteworthy. Throughout this thesis, we always write cl to denote the closure in the L-space (with respect to π, see Remark 2.18), rather than the closure in the localic part.

Lemma 4.16. *Let X be an* L*-space.*

(1) $\operatorname{cl} U \cap \operatorname{loc} X = U \cap \operatorname{loc} X$ *for each* $U \in \operatorname{OpUp}(X)$.

(2) $\operatorname{cl} U \cap \operatorname{loc} X = U$ *for each open subset* U *of* $\operatorname{loc} X$.

If X in addition is L*-spatial, then*

(3) $\operatorname{cl} U \cap \operatorname{cl} V = \operatorname{cl}(U \cap V)$ *for all open subsets* U *and* V *of* $\operatorname{loc} X$.

Proof. (1) The inclusion $U \cap \operatorname{loc} X \subseteq \operatorname{cl} U \cap \operatorname{loc} X$ is obvious. For the reverse inclusion, let $y \in \operatorname{cl} U \cap \operatorname{loc} X$. Since $y \in \operatorname{loc} X$, it follows that ${\downarrow} y$ is open. Thus, ${\downarrow} y \cap U \neq \varnothing$, meaning that there exists $x \in U$ with $x \leq y$. Since U is an upset, we conclude that $y \in U$, so $y \in U \cap \operatorname{loc} X$.

(2) Since U is open in $\operatorname{loc} X$, there exists $V \in \operatorname{ClopUp}(X)$ such that $U = V \cap \operatorname{loc} X$. This gives $\operatorname{cl} U \subseteq V$, and hence

$$U \subseteq \operatorname{cl} U \cap \operatorname{loc} X \subseteq V \cap \operatorname{loc} X = U.$$

(3) Let U and V be open subsets of $\operatorname{loc} X$. Then there exist $U', V' \in \operatorname{ClopUp}(X)$ such that $U = U' \cap \operatorname{loc} X$ and $V = V' \cap \operatorname{loc} X$. Since X is L-spatial, $\operatorname{loc} X$ is dense in X, so $U' = \operatorname{cl} U$ and $V' = \operatorname{cl} V$. Therefore, since $U' \cap V'$ is clopen in X, we obtain

$$U' \cap V' = \operatorname{cl}((U' \cap V') \cap \operatorname{loc} X) = \operatorname{cl}((U' \cap \operatorname{loc} X) \cap (V' \cap \operatorname{loc} X)) = \operatorname{cl}(U \cap V).$$

Thus, $\operatorname{cl} U \cap \operatorname{cl} V = U' \cap V' = \operatorname{cl}(U \cap V)$. $\square$

The next result is key to lifting a continuous map between the localic parts of two L-spaces to an L-morphism between the L-spaces. Since prime filters of clopen upsets correspond uniquely to points in the space, identifying a prime filter in the clopen upsets of the codomain allows us to find a point.

Lemma 4.17. *Let X and Y be spatial* L*-spaces,* $g : \operatorname{loc} X \to \operatorname{loc} Y$ *a continuous map, and* $x \in X$*. Then* $\mathscr{P}_x := \{U \in \operatorname{ClopUp}(Y) \mid x \in \operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y)\}$ *is a prime filter in* $\operatorname{ClopUp}(Y)$*.*

Proof. It is clear that $\mathscr{P}_x$ is an upset. To see that $\mathscr{P}_x$ is a filter, let $U, V \in \mathscr{P}_x$. Then $x \in \operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y)$ and $x \in \operatorname{cl} g^{-1}(V \cap \operatorname{loc} Y)$. Since $U \cap \operatorname{loc} Y$ and $V \cap \operatorname{loc} Y$ are open in $\operatorname{loc} Y$ and g is continuous, it follows that $g^{-1}(U \cap \operatorname{loc} Y)$ and $g^{-1}(V \cap \operatorname{loc} Y)$ are open in $\operatorname{loc} X$. Applying Lemma 4.16(3), we obtain

$$\begin{aligned} x \in \operatorname{cl}(g^{-1}(U \cap \operatorname{loc} Y)) \cap \operatorname{cl}(g^{-1}(V \cap \operatorname{loc} Y)) &= \operatorname{cl}(g^{-1}(U \cap \operatorname{loc} Y) \cap g^{-1}(V \cap \operatorname{loc} Y)) \\ &= \operatorname{cl}(g^{-1}((U \cap V) \cap \operatorname{loc} Y)). \end{aligned}$$

Thus, $U \cap V \in \mathscr{P}_x$, proving that $\mathscr{P}_x$ is a filter.

To see that it is prime, suppose $U \cup V \in \mathscr{P}_x$. Then $x \in \operatorname{cl} g^{-1}((U \cup V) \cap \operatorname{loc} Y)$. Since, cl commutes with finite unions, and g^{-1} commutes with unions and intersections, we obtain

$$\operatorname{cl} g^{-1}((U \cup V) \cap \operatorname{loc} Y) = \operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y) \cup \operatorname{cl} g^{-1}(V \cap \operatorname{loc} Y).$$

Therefore, $x \in \operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y)$ or $x \in \operatorname{cl} g^{-1}(V \cap \operatorname{loc} Y)$, meaning that either $U \in \mathscr{P}_x$ or $V \in \mathscr{P}_x$. Thus, $\mathscr{P}_x$ is a prime filter. □

The following proposition shows that any continuous map between localic parts of spatial L-spaces extends uniquely to an L-morphism between the entire L-spaces.

Proposition 4.18. *Suppose that X and Y are spatial* L*-spaces and $g : \operatorname{loc} X \to \operatorname{loc} Y$ is a continuous map. Then there is an* L*-morphism $f : X \to Y$ which extends g.*

Proof. Let $x \in X$. By Lemma 4.17, the set

$$\mathscr{P}_x = \{U \in \operatorname{ClopUp}(Y) \mid x \in \operatorname{cl}(g^{-1}(U \cap \operatorname{loc} Y))\}$$

is a prime filter of ClopUp(X). By Proposition 2.10, we have $\bigcap \mathscr{P}_x = {\uparrow} z_x$ for a unique $z_x \in Y$. Define $f : X \to Y$ by $f(x) = z_x$ for each $x \in X$. It is clear that f is well defined.

To see that f extends g, suppose $y \in \operatorname{loc} X$. Then

$$\begin{aligned}
{\uparrow} g(y) &= \bigcap \{U \in \operatorname{ClopUp}(Y) \mid g(y) \in U\} \\
&= \bigcap \{U \in \operatorname{ClopUp}(Y) \mid g(y) \in U \cap \operatorname{loc} Y\} \\
&= \bigcap \{U \in \operatorname{ClopUp}(Y) \mid y \in g^{-1}(U \cap \operatorname{loc} Y)\} \\
&= \bigcap \{U \in \operatorname{ClopUp}(Y) \mid y \in \operatorname{cl}[g^{-1}(U \cap \operatorname{loc} Y)]\} = \bigcap \mathscr{P}_y,
\end{aligned}$$

where the second-to-last equality follows from Lemma 4.16(2). Thus, by definition of f, we have $f(y) = g(y)$.

To see that f is continuous, let $U \in \operatorname{ClopUp}(Y)$. Then $U \cap \operatorname{loc} Y$ is open in $\operatorname{loc} Y$. Since g is continuous, $g^{-1}(U \cap \operatorname{loc} Y)$ is open in $\operatorname{loc} X$, so $\operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y) \in \operatorname{ClopUp}(X)$ (because X is L-spatial). Since $\operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y) = f^{-1}(U)$, it follows that

$$\begin{aligned}
x \in \operatorname{cl} g^{-1}(U \cap \operatorname{loc} Y) &\iff U \in \mathscr{P}_x \iff \bigcap \mathscr{P}_x \subseteq U \iff {\uparrow} f(x) \subseteq U \\
&\iff f(x) \in U \iff x \in f^{-1}(U),
\end{aligned}$$

where in the second equivalence we use that U is clopen, and thus compact. This shows that $f^{-1}(U) \in \operatorname{ClopUp}(X)$. Since clopen downsets are complements of clopen upsets, we also have that $f^{-1}(D) \in \mathrm{ClopDn}(X)$ for each $D \in \mathrm{ClopDn}(Y)$. Therefore, f is continuous as clopen upsets and clopen downsets form a subbasis of Y (see Lemma 2.8(1)).

To see that f is order-preserving, note that since $\mathrm{cl}\, g^{-1}(U \cap \mathrm{loc}\, Y) = f^{-1}(U)$ is an upset, the relation $x \leq z$ implies that $\mathscr{P}_x \subseteq \mathscr{P}_z$. Therefore, $\bigcap \mathscr{P}_z \subseteq \bigcap \mathscr{P}_x$, so $f(x) \leq f(z)$.

It remains to show that $\mathrm{cl}\, f^{-1}(U) = f^{-1}\,\mathrm{cl}\, U$ for each $U \in \mathrm{OpUp}(Y)$. The left-to-right inclusion follows from the continuity of f. For the right-to-left inclusion, let $x \in f^{-1}(\mathrm{cl}\, U)$. Then $f(x) \in \mathrm{cl}\, U$, so $\uparrow f(x) \subseteq \mathrm{cl}\, U$ by Lemma 3.4(3). Consequently, $\bigcap \mathscr{P}_x \subseteq \mathrm{cl}\, U$. Since $\mathrm{cl}\, U$ is open by (I.2), and $\mathscr{P}_x$ is a filter, compactness ensures the existence of $V \in \mathscr{P}_x$ such that $V \subseteq \mathrm{cl}\, U$. Since the former means

$$x \in \mathrm{cl}\, g^{-1}(V \cap \mathrm{loc}\, Y) = \mathrm{cl}\, f^{-1}(V \cap \mathrm{loc}\, Y),$$

combining this with $V \subseteq \mathrm{cl}\, U$ and applying Lemma 4.16(1), we obtain

$$x \in \mathrm{cl}\, f^{-1}(V \cap \mathrm{loc}\, Y) \subseteq \mathrm{cl}(f^{-1}(\mathrm{cl}\, U \cap \mathrm{loc}\, Y)) = \mathrm{cl}\, f^{-1}(U \cap \mathrm{loc}\, Y) \subseteq \mathrm{cl}\, f^{-1}(U).$$

Thus, f is an L-morphism. □

Having established that $\mathscr{L}oc$ is essentially surjective, we can now prove that it is full and faithful.

Theorem 4.19. *$\mathscr{L}oc : \mathsf{SLPries} \to \mathsf{Sob}$ is full and faithful.*

Proof. To see that $\mathscr{L}oc$ is full, suppose $g : \mathrm{loc}\, X \to \mathrm{loc}\, Y$ is a continuous map. By Proposition 4.18, there exists an L-morphism $f : X \to Y$ that extends g. Thus, $\mathscr{L}oc(f) = g$, proving fullness.

To show that $\mathscr{L}oc$ is faithful, suppose $f, g : X \to Y$ are L-morphisms with $f \neq g$. Since $\mathrm{loc}\, X$ is a dense subset of X and Y is Hausdorff, f and g must be the unique extensions of their restrictions $\mathscr{L}oc(f)$ and $\mathscr{L}oc(g)$ to $\operatorname{loc} X$ (see, e.g., [Eng89, p. 70]). Consequently, $\mathscr{L}oc(f) \neq \mathscr{L}oc(g)$, proving faithfulness. $\square$

We thus conclude that $\mathscr{L}oc$ establishes an equivalence:

Corollary 4.20. $\mathsf{SLPries}$ *is equivalent to* Sob*.*

Proof. By Theorems 4.15 and 4.19, the functor $\mathscr{L}oc\colon \mathsf{SLPries} \to \mathsf{Sob}$ is essentially surjective, full, and faithful. Therefore, $\mathscr{L}oc$ is an equivalence of categories (see, e.g., [ML98, p. 93]). $\square$

Combining Theorem 4.7 and Corollary 4.20 provides an alternative proof of the well-known result from Section 1 that SFrm is dually equivalent to Sob (see Theorem 1.10). The results of this section are summarized in the final two rows of Fig. 3, where we use the same notation as in the previous chapter. In Chapter III, we will refine this correspondence further, restricting it to obtain an alternative proof of Hofmann–Lawson duality.

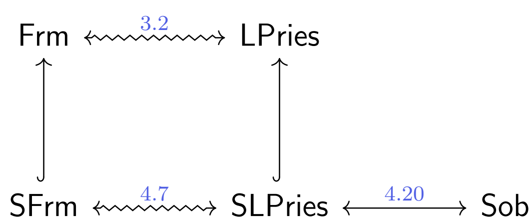

Figure 3: Restricting Pultr–Sichler duality to spatial frames results in the dual equivalence of SFrm and Sob through Priestley duality.

5 Priestley spaces of compact frames: Scott upsets and compactness

Having established the equivalence between $\mathsf{SLPries}$ and Sob via the functor $\mathscr{L}oc$, we now turn our attention to compact frames and their Priestley spaces. We focus on a characterization of compact elements in the language of Priestley spaces and describe the way-below relation in terms of Priestley duality. The main insight is that compactness of a frame can be understood through the localic part of its Priestley space. In particular, the compact elements of a frame correspond to clopen Scott upsets in the Priestley space, which are special closed upsets whose minimal elements belong to the localic part.

To formalize this, we introduce Scott upsets, which will play a central role in the upcoming results.

Definition 5.1. Let X be an L-space. We call $F \in \mathrm{ClUp}(X)$ a *Scott upset* if $\min F \subseteq \mathrm{loc}\, X$. We denote by $\mathrm{SUp}(X)$ the subposet of $\mathrm{ClUp}(X)$ consisting of Scott upsets.

Scott upsets are those closed upsets that have their minimal elements in $\mathrm{loc}\, X$ (see Fig. 4). Their importance lies, among other things, in the connection to compactness: as we will see in Section 6, they provide a description of Scott-open filters in the language of Priestley spaces.

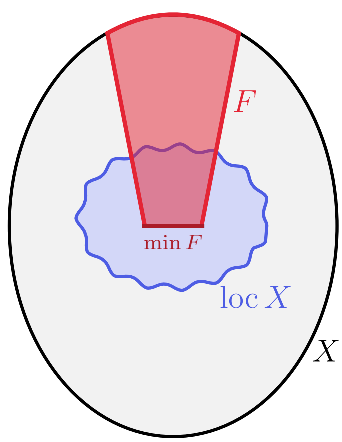

Figure 4: Scott upsets visually.

The next lemma provides an alternative characterization of Scott upsets, showing that they can also be described in terms of how they interact with the closure of open upsets.

Lemma 5.2. *Let X be an* L*-space and $F \in \mathrm{ClUp}(X)$. The following are equivalent:*

(1) $\min F \subseteq \mathrm{loc}\, X$.

(2) $F = {\uparrow}(F \cap \mathrm{loc}\, X)$.

(3) *For each $U \in \mathrm{OpUp}(X)$, from $F \subseteq \mathrm{cl}\, U$ it follows that $F \subseteq U$.*

Proof. (1)⇒(2) By Lemma 2.9(2), $F = {\uparrow}\min F$. Since $\min F \subseteq F \cap \mathrm{loc}\, X$ by (1),

$$F = {\uparrow}\min F \subseteq {\uparrow}(F \cap \mathrm{loc}\, X) \subseteq F,$$

where the last inclusion holds because F is an upset.

(2)⇒(3) Suppose $U \in \mathrm{OpUp}(X)$ such that $F \subseteq \mathrm{cl}\, U$. Then, applying Lemma 4.16(1), we obtain

$$F \cap \mathrm{loc}\, X \subseteq \mathrm{cl}\, U \cap \mathrm{loc}\, X = U \cap \mathrm{loc}\, X.$$

Therefore, $F \cap \mathrm{loc}\, X \subseteq U$. By (2), $F = {\uparrow}(F \cap \mathrm{loc}\, X) \subseteq U$ since U is an upset.

(3)⇒(1) Suppose towards a contradiction that there exists $y \in \min F \setminus \mathrm{loc}\, X$. Then ${\downarrow}y$ is not open, so $U := X \setminus {\downarrow}y$ is not closed. Consequently, ${\downarrow}y \cap \mathrm{cl}\, U \neq \varnothing$. Since ${\downarrow}y$ is a closed downset, it follows that $U \in \mathrm{OpUp}(X)$, and therefore $\mathrm{cl}\, U \in \mathrm{ClopUp}(X)$ by (I.2). Thus, $y \in \mathrm{cl}\, U$, and hence, by Lemma 2.9(2), $F = {\uparrow}\min F \subseteq \mathrm{cl}\, U$. Applying (3), we obtain $F \subseteq U$, contradicting $y \notin U$. Consequently, $\min F \subseteq \mathrm{loc}\, X$, completing the proof. □

It is worth noting that in the implication (3)⇒(1), we proved the following fact, which will be utilized in what follows.

Lemma 5.3. *Let X be an* L*-space. If $x \notin \mathrm{loc}\, X$, then $U := X \setminus {\downarrow}x$ is an open upset such that $x \in \mathrm{cl}\, U$.*

Remark 5.4. In [PS00], closed sets satisfying Lemma 5.2(3) are referred to as "L-compact" sets. This makes Scott upsets the upsets of L-compact sets.

Lemma 5.2(3) is closely related to the *way-below relation*, which we can express in terms of Priestley duality.

Definition 5.5 (see, e.g., [GHK+03, p. 49])**.** Let L be a frame and $a, b \in L$.

(1) We say *a is way below b*, written $a \ll b$, provided for each $S \subseteq L$, from $b \leq \bigvee S$ it follows that $a \leq \bigvee T$ for some finite $T \subseteq S$.

(2) The element a is said to be *compact* if $a \ll a$.

(3) The frame L is said to be *compact* if its top element 1 is compact.

Proposition 5.6. *Let L be a frame, X its Priestley space, and $a, b \in L$. Then $a \ll b$ iff*

$$\varphi(b) \subseteq \operatorname{cl} U \text{ implies } \varphi(a) \subseteq U \tag{II.1}$$

for each $U \in \operatorname{OpUp}(X)$.

Proof. First, suppose $a \ll b$ and $U \in \operatorname{OpUp}(X)$ such that $\varphi(b) \subseteq \operatorname{cl} U$. Since $U = \bigcup \varphi[S]$ for some $S \subseteq L$ (see Lemma 2.8(2)), it follows that $\varphi(b) \subseteq \operatorname{cl} \bigcup S$. By (I.3), we obtain that $b \leq \bigvee S$. Since $a \ll b$, there exists a finite subset $T \subseteq S$ such that $a \leq \bigvee T \subseteq S$. Thus,

$$\varphi(a) \subseteq \bigcup \varphi[T] \subseteq U.$$

Conversely, suppose (II.1) holds for each $U \in \operatorname{OpUp}(X)$. Let $b \leq \bigvee S$ for some $S \subseteq L$. By (I.3), $\varphi(b) \subseteq \operatorname{cl} \bigcup \varphi[S]$. Therefore, $\varphi(a) \subseteq \bigcup \varphi[S]$ by (II.1). By compactness, there exists a finite subset $T \subseteq S$ such that $\varphi(a) \subseteq \bigcup \varphi[T]$. Thus, $a \leq \bigvee T$, which implies that $a \ll b$. $\square$

This leads to a natural definition of the way-below relation for subsets of an L-space.

Definition 5.7. Let X be an L-space. For subsets $Y, Z \subseteq X$, we write $Y \ll Z$ iff for each $U \in \mathrm{OpUp}(X)$, from $Z \subseteq \mathrm{cl}\, U$ it follows that $Y \subseteq U$.

Using this definition and Lemma 5.2(3), we see that Scott upsets are precisely the closed upsets F satisfying $F \ll F$. The next result establishes that compact elements of a frame correspond to clopen Scott upsets of its Priestley space.

Theorem 5.8. *Let L be a frame and X its Priestley space.*

(1) *$a \in L$ is compact iff $\varphi(a)$ is a Scott upset.*

(2) *L is compact iff $\min X \subseteq \mathrm{loc}\, X$.*

Proof. (1) This follows directly from Lemma 5.2 and Proposition 5.6.

(2) Since $\varphi(1) = X$, it follows from (1) that

$$L \text{ is compact} \iff 1 \text{ is compact} \iff \varphi(1) \text{ is a Scott upset} \iff \min X \subseteq \mathrm{loc}\, X. \qquad \square$$

Remark 5.9. While Theorem 5.8(2) is already known (see [BGJ16, Lem. 3.1]), the above proof is particularly short.

The final step in this section is to restrict Pultr–Sichler duality to compact frames.

Definition 5.10. An L-space X is L-*compact* if $\min X \subseteq \mathrm{loc}\, X$.

Let KFrm be the full subcategory of Frm consisting of compact frames and let $\mathsf{KLPries}$ be the full subcategory of LPries consisting of compact L-spaces.

Corollary 5.11. $\mathsf{KLPries}$ *is dually equivalent to* KFrm*.*

Proof. The result follows by restricting Theorem 3.2 to the subcategory of compact frames, using Theorem 5.8(2) to establish the equivalence. $\square$

We next connect compact L-spaces to compact sober spaces. Since compact frames are not spatial, we need to restrict to compact spatial frames to obtain a one-to-one correspondence between compact L-spaces and compact topological spaces.

Lemma 5.12. *Let X be an* L*-space. If X is* L*-compact, then $\operatorname{loc} X$ is compact. If in addition X is* L*-spatial, then the converse also holds.*

Proof. Suppose X is L-compact. Let $\mathcal{U}$ be an open cover of $\operatorname{loc} X$. For each $U' \in \mathcal{U}$, there exists $U \in \operatorname{ClopUp}(X)$ such that $U \cap \operatorname{loc} X = U'$. Since X is L-compact,

$$\min X \subseteq \operatorname{loc} X \subseteq \bigcup\{U \mid U' \in \mathcal{U}\}.$$

Therefore, $X = {\uparrow}\min X \subseteq \bigcup\{U \mid U' \in \mathcal{U}\}$ because the latter is an upset. Since X is compact, there exist $U_1', \dots, U_n' \in \mathcal{U}$ such that $X \subseteq U_1 \cup \cdots \cup U_n$. Thus, we have

$$\operatorname{loc} X \subseteq (U_1 \cup \cdots \cup U_n) \cap \operatorname{loc} X = U_1' \cup \cdots \cup U_n',$$

which shows that $\operatorname{loc} X$ is compact.

Conversely, suppose X is not L-compact. Then there exists $x \in \min(X) \setminus \operatorname{loc} X$. By Lemma 5.3, $x \in \operatorname{cl} U$, where $U = X \setminus {\downarrow}x \in \operatorname{OpUp}(X)$. Since U is an open upset, it follows from Lemma 2.8(2) that $U = \bigcup\{V \in \operatorname{ClopUp}(X) \mid x \notin V\}$. Therefore,

$$X = \operatorname{cl} U = \operatorname{cl} \bigcup\{V \in \operatorname{ClopUp}(X) \mid x \notin V\},$$

and hence $\operatorname{loc} X \subseteq \bigcup\{V \in \operatorname{ClopUp}(X) \mid x \notin V\}$ by Lemma 4.16(1). If $\operatorname{loc} X$ were compact, there would exist $V \in \operatorname{ClopUp}(X)$ such that $x \notin V$ and $\operatorname{loc} X \subseteq V$. Since X is L-spatial, this would imply that $x \in X = \operatorname{cl} \operatorname{loc} X \subseteq V$, a contradiction. Thus, $\operatorname{loc} X$ is not compact. $\square$

Theorem 5.13. *For a spatial frame L and its Priestley space X, the following are equivalent:*

(1) *L is compact.*

(2) *X is* L*-compact.*

(3) $\operatorname{loc} X$ *is compact.*

Proof. (1)⇔(2) This is Theorem 5.8(2).

(2)⇔(3) This follows from Lemma 5.12 since X is L-spatial by Theorem 4.4. □

Let KSFrm be the full subcategory of SFrm consisting of compact spatial frames, KSLPries the full subcategory of SLPries consisting of compact SL-spaces, and KSob the full subcategory of Sob consisting of compact sober spaces.

Corollary 5.14.

(1) KSLPries *is dually equivalent to* KSFrm*.*

(2) KSLPries *is equivalent to* KSob*.*

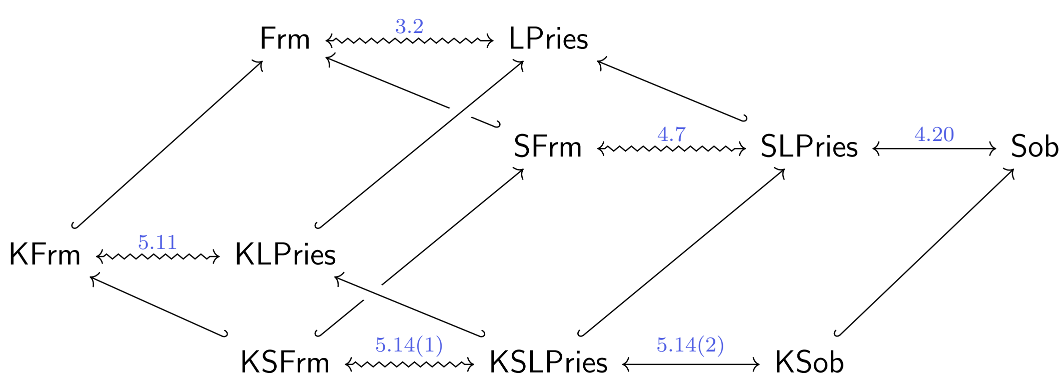

Figure 5: Equivalences and dual equivalences among categories of compact frames, compact L-spaces, and compact topological spaces.

The results of this section are summarized in Fig. 5. They clarify how compactness of a frame is linked to its space of points. Through Priestley duality, compact elements of a frame correspond to clopen Scott upsets, which are defined in terms of their minimal elements. Since these minimal elements belong to the localic part, the presence of compact elements in the frame is determined by the structure of the localic part. However, the localic part coincides with the space of points of the frame (see Lemma 4.9(2)). Thus, compactness at the frame level is not merely reflected in the space of points, it is a fundamental property of it. The dual equivalence between compact frames and compact L-spaces further reinforces this connection, showing that compactness of a frame translates to its Priestley space being a Scott upset.

In the next section, we will see how Scott upsets play a central role in proving the Hofmann–Mislove Theorem, further demonstrating their significance.

6 Scott upsets and their role in Hofmann–Mislove

We begin the section by describing the equivalence of the Hofmann–Mislove Theorem and Theorem 2.11 in the setting of spectral spaces. The following definition is standard (see, e.g., [GHK+03, p. 134]).

Definition 6.1. Let L be a frame. Then $F \in \mathrm{Filt}(L)$ is called *Scott-open* if $\bigvee S \in F$ implies $\bigvee T \in F$ for some finite $T \subseteq S$. Let $\mathrm{OFilt}(L)$ be the poset of Scott-open filters of L, ordered by reverse inclusion.

For a topological space X, we denote by $\mathrm{KSat}(X)$ the poset of compact saturated subsets ordered by inclusion.

Remark 6.2. It is more customary to order $\mathrm{OFilt}(L)$ by inclusion and $\mathrm{KSat}(X)$ by reverse inclusion (see [GHK+03, Sec. II-1]). Our ordering is motivated by how we ordered posets of filters and closed upsets in Section 2.

Theorem 6.3 (Hofmann–Mislove [HM81])**.** *Let X be a sober space. Then* $\mathrm{KSat}(X)$ *is isomorphic to* $\mathrm{OFilt}(\Omega(L))$*.*

As observed in [BBGK10], when X is a spectral space, the Hofmann–Mislove Theorem and Theorem 2.11 are equivalent results:

Remark 6.4.

(1) Let D be a bounded distributive lattice and L be its frame of ideals. Then L is a coherent frame (see, e.g. [Joh82, p. 64]), meaning that its compact elements form a bounded sublattice that join-generates L. Moreover, sending D to L defines a covariant functor that establishes an equivalence (see, e.g. [Joh82, p. 65]) between DLat and the category CohFrm of coherent frames and coherent frame homorphisms (i.e., frame homomorphisms that preserve compact elements, see Definition 12.6(1)). Under this equivalence, the posets $\mathrm{Filt}(D)$ and $\mathrm{OFilt}(L)$ are isomorphic.

(2) Let X be the Priestley space of D. By Theorem 2.11, the poset $\mathrm{Filt}(D)$ is isomorphic to $\mathrm{ClUp}(X)$, which in turn is isomorphic to $\mathrm{KSat}(X_\tau)$, where X_τ is the spectral space corresponding to X (see Theorem 2.17). Since every spectral space arises this way, it follows that Hofmann–Mislove Theorem for spectral spaces is equivalent to Theorem 2.11.

We now extend Remark 6.4(1) beyond spectral spaces by using Pultr–Sichler duality to prove the Hofmann–Mislove Theorem for arbitrary frames and then specialize to sober spaces. The key to proving the theorem is the one-to-one correspondence between Scott-open filters of the frame and Scott upsets of its Priestley space.

If F is a filter of a frame L, we recall from Theorem 2.11 that its corresponding closed upset is given by $K_F = \bigcap\{\varphi(a) \mid a \in F\}$. The next lemma establishes a key connection between Scott-open filters and Scott upsets.

Lemma 6.5. *Let L be a frame and X its Priestley space. For a filter $F \subseteq L$, F is Scott-open iff K_F is a Scott upset.*

Proof. First, suppose K_F is a Scott upset and $\bigvee S \in F$ for some $S \subseteq L$. Then

$$K_F \subseteq \varphi\left(\bigvee S\right) = \operatorname{cl}\bigcup \varphi[S].$$

By Lemma 5.2, $K_F \subseteq \bigcup \varphi[S]$. Since K_F is closed, it is compact, so there exists a finite subset $T \subseteq S$ such that $K_F \subseteq \bigcup \varphi[T] = \varphi(\bigvee T)$. Thus, $\bigvee T \in F$, and hence F is Scott-open.

Conversely, suppose K_F is not a Scott upset, so there exists $x \in \min K_F \setminus \operatorname{loc} X$. By Lemma 5.3, $x \in \operatorname{cl} U_x$, where $U_x = X \setminus {\downarrow}x \in \operatorname{OpUp}(X)$. Define $S \coloneqq \{s \in L \mid \varphi(s) \subseteq U_x\}$. Then, by Lemma 2.8(2), $U_x = \bigcup \varphi[S]$. Applying (I.3), we get

$$x \in \operatorname{cl} U_x = \operatorname{cl}\bigcup \varphi[S] = \varphi\left(\bigvee S\right).$$

Since $(\min K_F) \cap {\downarrow}x = \{x\}$ and $x \in \varphi(\bigvee S)$, it follows that $K_F = {\uparrow}\min K_F \subseteq \varphi(\bigvee S)$ (see Lemma 2.9(2)). Thus, $\bigvee S \in F$ (see Theorem 2.11 and (I.3)).

On the other hand, for each $s \in S$, we have $x \notin \varphi(s)$. Since for each finite $T \subseteq S$, we have $\varphi(\bigvee T) = \bigcup \varphi[T]$, it follows that $x \notin \varphi(\bigvee T)$. Thus, $K_F \not\subseteq \varphi(\bigvee T)$, which implies that $\bigvee T \notin F$ for all finite $T \subseteq S$. Consequently, F is not Scott-open. $\square$

As as a consequence, we establish an isomorphism between the poset of Scott-open filters of a frame and the poset of Scott upsets of its Priestley space.

Theorem 6.6. *Let L be a frame and X its Priestley space. Then* $\mathrm{OFilt}(L)$ *is isomorphic to* $\mathrm{SUp}(X)$.

Proof. By Theorem 2.11, the poset $\mathrm{Filt}(L)$ is isomorphic to $\mathrm{ClUp}(X)$. By Lemma 6.5, this isomorphism restricts to an isomorphism between $\mathrm{OFilt}(L)$ and $\mathrm{SUp}(X)$. □

The following theorem is the Priestley analogue of the Hofmann–Mislove Theorem. It plays a crucial role in proving Hofmann–Lawson duality via Priestley duality (see Section 8).

Theorem 6.7. *Let X be an* L*-space. Then* $\mathrm{SUp}(X)$ *is isomorphic to* $\mathrm{KSat}(\mathrm{loc}\, X)$. *This isomorphism is established by the maps:*

$$F \mapsto F \cap \mathrm{loc}\, X \qquad \textit{for } F \in \mathrm{SUp}(X),$$

$$Q \mapsto {\uparrow} Q \qquad \textit{for } Q \in \mathrm{KSat}(\mathrm{loc}\, X).$$

Proof. Define $f\colon \mathrm{SUp}(X) \to \mathrm{KSat}(\mathrm{loc}\, X)$ by $f(K) = K \cap \mathrm{loc}\, X$. We first verify that f is well defined. By [ABMZ20, Lem. 5.3], the specialization order on $\mathrm{loc}\, X$ is the restriction of the partial order on X to $\mathrm{loc}\, X$. Since K is an upset in X, it follows that $K \cap \mathrm{loc}\, X$ is saturated in $\mathrm{loc}\, X$.

To show compactness, suppose $K \cap \mathrm{loc}\, X \subseteq \bigcup \zeta(a_i)$. Applying Lemma 4.9(1) yields

$$\bigcup \zeta(a_i) = \bigcup (\mathrm{loc}\, X \cap \varphi(a_i)) = \mathrm{loc}\, X \cap \bigcup \varphi(a_i).$$

Thus, $K \cap \mathrm{loc}\, X \subseteq \bigcup \varphi(a_i)$. Since K is a Scott upset, we have $\min K \subseteq \mathrm{loc}\, X$, so $\min K \subseteq \bigcup \varphi(a_i)$. Applying Lemma 2.9(2), $K = {\uparrow} \min K \subseteq \bigcup \varphi(a_i)$. Since K is compact in X, there exist $a_{i_1}, \dots, a_{i_n}$ such that $K \subseteq \varphi(a_{i_1}) \cup \cdots \cup \varphi(a_{i_n})$. Consequently, $K \cap \mathrm{loc}\, X \subseteq \zeta(a_{i_1}) \cup \cdots \cup \zeta(a_{i_n})$, which shows that $K \cap \mathrm{loc}\, X$ is compact in $\mathrm{loc}\, X$. Hence, f is well defined, and it clearly preserves $\subseteq$.

Next, define $g\colon \mathrm{KSat}(\mathrm{loc}\, X) \to \mathrm{SUp}(X)$ by $g(Q) = {\uparrow}Q$. To show that ${\uparrow}Q$ is a closed upset, let $x \notin {\uparrow}Q$. Then $y \not\leq x$ for all $y \in Q$. By the Priestley separation axiom (see (I.1)), for each $y \in Q$, there exists $U_y \in \mathrm{ClopUp}(X)$ such that $y \in U_y$ and $x \notin U_y$. Therefore, $Q \subseteq \bigcup_{y \in Q} U_y$. Since $U_y \cap \mathrm{loc}\, X$ is open in $\mathrm{loc}\, X$, Q is compact in $\mathrm{loc}\, X$, and a finite union of clopen upsets of X is a clopen upset of X, we can conclude that there exists $U \in \mathrm{ClopUp}(X)$ such that $Q \subseteq U$ and $x \notin U$. Since U is an upset in X, it follows that ${\uparrow}Q \subseteq U$. Thus, ${\uparrow}Q$ is the intersection of clopen upsets of X containing ${\uparrow}Q$, and hence ${\uparrow}Q$ is a closed upset. Therefore, g is well defined, and it clearly preserves $\subseteq$.

Finally, we show that f and g are inverses of each other. If K is a Scott upset of X, then

$$gf(K) = {\uparrow}(K \cap \mathrm{loc}\, X) = K$$

by Lemma 5.2. Similarly, if Q is compact saturated in $\mathrm{loc}\, X$, then

$$fg(Q) = {\uparrow}Q \cap \mathrm{loc}\, X = Q.$$

Thus, f and g are order-preserving maps that are inverses of each other, proving that $\mathrm{SUp}(X)$ is isomorphic to $\mathrm{KSat}(\mathrm{loc}\, X)$. □

The Hofmann–Mislove Theorem follows immediately from Theorems 6.6 and 6.7. However, the result extends beyond the sober setting: we have shown that for an arbitrary frame L, the poset of Scott-open filters $\mathrm{OFilt}(L)$ is isomorphic to the poset of compact saturated sets of $\mathrm{pt}(L)$. This general form of the theorem appears in [Vic89, Thm. 8.2.5].

Corollary 6.8 (Hofmann–Mislove)**.**

(1) *If L is a frame, then* $\mathrm{OFilt}(L)$ *is isomorphic to* $\mathrm{KSat}(\mathrm{pt}(L))$*.*

(2) *If X is a sober space, then* $\mathrm{OFilt}(\Omega(X))$ *is isomorphic to* $\mathrm{KSat}(X)$*.*

Proof. (1) Let X be the Priestley space of L. By applying Theorems 6.6 and 6.7, we have that $\mathrm{OFilt}(L)$ is isomorphic to $\mathrm{KSat}(\mathrm{loc}\, X)$. The result follows as $\mathrm{loc}\, X$ is homeomorphic to $\mathrm{pt}(L)$ (see Lemma 4.9(2)).

(2) Since X is sober, it is homeomorphic to $\mathrm{pt}(\Omega(X))$ (see Theorem 1.10). Applying (1) completes the proof. □

The following corollary describes the relationship between Scott-open filters and completely prime filters of a frame. Corollary 6.9(2) appears in [Ban81, Lem. 3].

Corollary 6.9. *Let L be a frame and X its Priestley space.*

(1) *A Scott-open filter F is completely prime iff $\min K_F$ is a singleton.*

(2) *Every Scott-open filter of L is an intersection of completely prime filters of L.*

Proof. (1) It is well known (see, e.g., [GHK+03, p. 414]) and easy to see that a Scott-open filter is completely prime iff it is prime. Thus, F is completely prime iff $\min K_F$ is a singleton (see, e.g., [BBGK10, Cor. 6.7]).

(2) Let F be a Scott-open filter and suppose $a \notin F$. Then $K_F \not\subseteq \varphi(a)$, so there exists $y \in \min K_F$ with $y \notin \varphi(a)$. By Lemma 6.5, K_F is a Scott upset, which implies that $y \in \mathrm{loc}\, X$. Thus, by Lemma 4.1, y is completely prime. Moreover, since $y \in K_F$, it follows that $F \subseteq y$. Furthermore, since $y \notin \varphi(a)$, we have $a \notin y$. Thus, there exists a completely prime filter y that contains F but does not contain a. Consequently, F is the intersection of completely prime filters of L that contain F. □

Remark 6.10. The Hofmann–Mislove Theorem is traditionally proved using Zorn's Lemma. Our proof also depends on Zorn's Lemma as it is used in Lemma 2.9(2). However, this reliance can be avoided by defining Scott upsets in terms of either (2) or (3) of Lemma 5.2.

Under the assumption of Zorn's Lemma, these formulations are equivalent to being a Scott upset (as shown in Lemma 5.2). Consequently, when working through Priestley duality, the Hofmann–Mislove Theorem can be derived using only the Prime Ideal Theorem, which is strictly weaker than Zorn's Lemma and the Axiom of Choice. Moreover, [Ern18, Thm. 3] establishes that the Hofmann–Mislove Theorem is equivalent to the Prime Ideal Theorem, meaning that this assumption cannot be avoided.

In the next chapter of the thesis, we will restrict the equivalences between SFrm, SLPries, and Sob using Theorem 6.7. The importance of this theorem lies in its role as the Priestley analogue of the Hofmann–Mislove Theorem.

Chapter III

A Priestley journey from Hofmann–Lawson to Isbell

The well-known equivalence between Sob and SFrm (see Theorem 1.10) restricts to several prominent duality results in pointfree topology. Notably, these include the following:

- **Hofmann–Lawson duality**: The category ConFrm of continuous frames with proper frame homomorphisms is dually equivalent to the category LKSob of locally compact spaces with proper continuous maps [HL78].
- **Dual equivalences for stably continuous frames**: The full subcategory StCFrm of ConFrm consisting of stably continuous frames is dually equivalent to the full subcategory StLKSp of LKSob consisting of stably locally compact spaces (see, e.g., [Ban81, GK77, Joh81, Sim82]). This further restricts to a dual equivalence between the full subcategories StKFrm of stably compact frames and StKSp of stably compact spaces.
- **Isbell duality**: The full subcategory KRFrm of Frm consisting of compact regular frames is dually equivalent to the full subcategory KHaus of Top consisting of compact Hausdorff spaces [Isb72] (see also [BM80, Joh82]).

Since every frame homomorphism between compact regular frames is proper, KRFrm forms a full subcategory of StKFrm. Similarly, KHaus is a full subcategory of StKSp. We thus arrive at the diagram in Fig. 6, where we use the same notation as in the previous chapters (i.e., A $\leftrightsquigarrow$ B indicates that A is dually equivalent to B and A $\hookrightarrow$ B that A is a full subcategory of B).

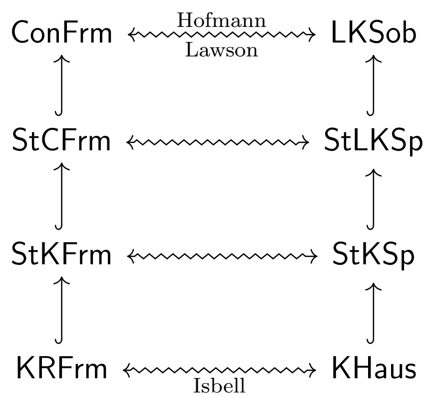

Figure 6: Correspondence between various categories of continuous frames and locally compact sober spaces.

This chapter provides a dual description of the categories ConFrm, StCFrm, StKFrm, and KRFrm in the language of Priestley spaces. These descriptions lead to alternative proofs of the dual equivalences mentioned above while also offering new insights into these classic results from the perspective of Priestley duality. Specifically, we characterize continuity, stability, and regularity of a frame in terms of special maps on the clopen upsets of its Priestley space, which we term *kernels*. As we will see, kernels provide a powerful and systematic framework for rigorously describing properties of frames.

Beyond reproving these results, this approach establishes new subcategories of Priestley spaces that are equivalent to important categories of topological spaces such as LKSob, StLKSp, StKSp, and KHaus. We believe that results of this nature can foster further cross-fertilization between these branches of mathematics.

The chapter is organized as follows. In Section 7, we recall the relevant definitions and introduce the key categories of locally compact sober spaces and continuous frames, along with the duality results from Hofmann–Lawson to Isbell. Section 8 introduces a kernel associated with continuity and uses it to characterize continuous frames in terms of Priestley spaces. Then we connect the associated Priestley spaces with locally compact sober spaces, leading to a new proof of Hofmann–Lawson duality. In Section 9, we derive the duality between stably continuous frames and stably locally compact spaces by describing stability via kernels. This also provides a new proof of the duality between stably compact frames and stably compact spaces. Finally, Section 10 describes regularity in the language of Priestley spaces by introducing a kernel associated with regularity, thus offering an alternative proof of Isbell duality.

7 Locally compact sober spaces and continuous frames

In this section, we introduce the relevant categories of locally compact sober spaces and continuous frames. Two relations on frames are particularly important to us: the *way-below relation* $\ll$ (see Section 5) and the *well-inside relation* $\prec$ (see, e.g., [Joh82, p. 80]). Recall that for a frame L and an element $a \in L$, the *pseudocomplement* of a is defined by

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}.$$

We say that a is *well inside* b, written $a \prec b$, provided $a^* \vee b = 1$.

Definition 7.1. Let L be a frame.

(1) L is *continuous* if $a = \bigvee \{b \in L \mid b \ll a\}$ for all $a \in L$.

(2) L is *regular* if $a = \bigvee \{b \in L \mid b \prec a\}$ for all $a \in L$.

Each frame homomorphism $h\colon L \to M$ preserves the well-inside relation, meaning that $a \prec b$ implies $h(a) \prec h(b)$, but may not preserve the way-below relation.

Definition 7.2.

(1) A frame homomorphism $h\colon L \to M$ is *proper* provided it preserves the way-below relation, meaning that $a \ll b$ implies $h(a) \ll h(b)$.

(2) Let ConFrm be the category of continuous frames and proper frame homomorphisms between them.

Recall from Section 5 that an element a of a frame L is compact if $a \ll a$, and that L is compact if its top element is compact. The way-below relation $\ll$ is said to be *stable* if $a \ll b$ and $a \ll c$ imply $a \ll b \wedge c$ for all $a, b, c \in L$.

Definition 7.3.

(1) (see, e.g., [GHK+03, p. 488]) A frame L is *stably continuous* if L is continuous and $\ll$ is stable. Let StCFrm be the full subcategory of ConFrm consisting of stably continuous frames.

(2) (see, e.g., [GHK+03, p. 488]) A frame L is *stably compact* if L is compact and stably continuous. Let StKFrm be the full subcategory of StCFrm consisting of stably compact frames.

(3) (see, e.g., [PP12, p. 133]) Let KRFrm be the full subcategory of Frm consisting of compact regular frames.

If L is compact, then $a \prec b$ implies $a \ll b$, and if L is regular, then $a \ll b$ implies $a \prec b$ (see, e.g., [PP12, Lem. 5.2.1]). Therefore, in a compact regular frame, the way-below and well-inside relations coincide. Since $a \prec b$ and $a \prec c$ imply $a \prec b \wedge c$, it follows that every compact regular frame is stably compact. Consequently, KRFrm is a full subcategory of StKFrm.

An overview of the categories of frames defined above is given in the conclusions (see Table 2). We now turn our attention to the categories of spaces that correspond to these categories of frames.

Recall that a set is saturated if it is an intersection of open sets, and the specialization preorder on a topological space is defined by $x \leq y$ iff $x \in \mathrm{cl}\{y\}$ (see Remark 2.18).

Definition 7.4.

(1) A topological space X is *locally compact* if for each open set U and $x \in U$, there exist an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$.

(2) A continuous map $f\colon X \to Y$ between topological spaces is *proper* if

  (i) ${\downarrow} f(A)$ is closed for each closed set $A \subseteq X$, where $\downarrow$ is the downset in the specialization preorder on X.

  (ii) $f^{-1}(B)$ is compact for each compact saturated set $B \subseteq Y$.

(3) Let LKSob be the category of locally compact sober spaces and proper continuous maps between them.

The following definitions are well known; see, e.g., [GHK+03]:

Definition 7.5.

(1) The space X is *coherent* if the intersection of two compact saturated sets is again compact.

(2) A space X is *stably locally compact* if it is locally compact, sober, and coherent. Let StLKSp be the full subcategory of LKSob consisting of stably locally compact spaces.

(3) A space X is *stably compact* if it is compact and stably locally compact. Let StKSp be the full subcategory of StLKSp consisting of stably compact spaces.

(4) Let KHaus be the full subcategory of Sob consisting of compact Hausdorff spaces.

Remark 7.6.

(1) By [GHK+03, Lem. VI-6.21], if X is sober and Y is locally compact, then condition (i) follows from condition (ii) in Definition 7.4(2).

(2) In compact Hausdorff spaces, the specialization order is the identity. Hence, compact saturated sets are simply closed sets. Therefore, since every compact Hausdorff space is sober and locally compact, KHaus is a full subcategory of StKSp.

An overview of the categories of topological spaces defined above is given in the conclusions (see Table 3).

Restricting the dual equivalence of SFrm and Sob (see Theorem 1.10) yields the following well-known duality results:

Theorem 7.7.

(1) (Hofmann–Lawson duality) ConFrm *is dually equivalent to* LKSob.

(2) StCFrm *is dually equivalent to* StLKSp.

(3) StKFrm *is dually equivalent to* StKSp.

(4) (Isbell duality) KRFrm *is dually equivalent to* KHaus.

Hofmann–Lawson duality was established in [HL78] (see also [GHK+03, Prop. V-5.20]). The dualities in (2) and (3) of Theorem 7.7 trace back to [GK77, Joh81, Sim82, Ban81] (see also [GHK+03, Thm. VI-7.4]). Isbell duality was given in [Isb72] (see also [BM80] and [Joh82, Sec. VII-4]). These results yield the diagram in Fig. 6.

In the following sections, we will give new proofs of these results through Priestley duality.

8 Priestley spaces of continuous frames: the kernel of Hofmann–Lawson

In this section, we provide a proof of Hofmann–Lawson duality using Priestley duality. To do so, we describe the Priestley spaces of continuous frames and characterize the Priestley duals of proper frame homomorphisms. We then restrict the dual equivalence of SLPries with SFrm, along with its equivalence with Sob, to the categories of continuous L-spaces, continuous frames, and locally compact sober spaces. This offers a new perspective on Hofmann–Lawson duality.

For an L-space X, we describe continuity in terms of a map $\mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$. Since this approach will also be used to characterize several classes of Priestley spaces of different frames, we introduce the following terminology.

Definition 8.1. Let X be an L-space.

(1) A map $\mathtt{ker}\colon \mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$ is called a *kernel* if

- $\mathtt{ker}\, U \subseteq U$ for each $U \in \mathrm{ClopUp}(X)$;
- $\mathtt{ker}$ is monotone.

(2) A kernel $\mathtt{ker}$ is *representative* if $\mathtt{ker}\, U$ is dense in U for each $U \in \mathrm{ClopUp}(X)$.

(3) A kernel $\mathtt{ker}$ is *stable* if $\mathtt{ker}\, U \cap \mathtt{ker}\, V = \mathtt{ker}(U \cap V)$ for all $U, V \in \mathrm{ClopUp}(X)$.

Let X be an L-space and $U, V \subseteq X$. Recall (see Definition 5.7) that $V \ll U$ if $U \subseteq \mathrm{cl}\, W$ implies $V \subseteq W$ for each $W \in \mathrm{OpUp}(X)$.

Definition 8.2. Suppose X is an L-space and $U \in \mathrm{ClopUp}(X)$. The *continuous part of* U is defined as

$$\mathtt{con}\, U = \bigcup \{V \in \mathrm{ClopUp}(X) \mid V \ll U\}.$$

If X is the Priestley space of a frame L and $U = \varphi(a)$ for some $a \in L$, we simply write $\texttt{con}(a)$ for $\texttt{con}\, U$.

Remark 8.3. The continuous part of a clopen upset U is referred to as the kernel of U in [BM23, BM25].

We will now show that $\texttt{con}$ is a kernel and relate it to the way-below relation $\ll$.

Lemma 8.4. *Let X be an* L*-space and $U, V \in \mathrm{ClopUp}(X)$.*

(1) *The map* $\texttt{con}$ *is a kernel.*

(2) $V \subseteq \texttt{con}\, U$ *iff* $V \ll U$*.*

(3) $U \subseteq \mathrm{cl}\, W$ *implies* $\texttt{con}\, U \subseteq W$ *for each* $W \in \mathrm{OpUp}(X)$*.*

(4) U *is a Scott upset iff* $\texttt{con}\, U = U$*.*

Moreover, if X is the Priestley space of a frame L and $a, b \in L$, then

(5) $a \ll b$ *iff* $\varphi(a) \ll \varphi(b)$ *iff* $\varphi(a) \subseteq \texttt{con}(b)$*.*

Proof. (1) It is immediate from the definition that $\texttt{con}\, U \in \mathrm{OpUp}(X)$. Also, $\texttt{con}\, U \subseteq U$ because $V \ll U$ implies $V \subseteq U$ (since $U \in \mathrm{OpUp}(X)$ and $\mathrm{cl}\, U = U$). To see that $\texttt{con}$ is monotone, let $U_1, U_2, V \in \mathrm{ClopUp}(X)$ with $U_1 \subseteq U_2$ and $V \ll U_1$. Suppose $W \in \mathrm{OpUp}(X)$ such that $U_2 \subseteq \mathrm{cl}\, W$. Then $U_1 \subseteq \mathrm{cl}\, W$, so $V \subseteq W$. Hence, $V \ll U_2$. Consequently, $\texttt{con}\, U_1 \subseteq \texttt{con}\, U_2$.

(2) The right-to-left implication is immediate from the definition. For the left-to-right implication, if $V \subseteq \texttt{con}\, U$ then by compactness and directedness there is a clopen upset $V' \ll U$ such that $V \subseteq V'$. Therefore, $V \ll U$.

(3) Suppose $U \subseteq \mathrm{cl}\, W$ and let $x \in \texttt{con}\, U$. Then there exists $V \in \mathrm{ClopUp}(X)$ with $x \in V \ll U$. Hence, $x \in V \subseteq W$.

(4) Suppose U is a Scott upset. Then $U \ll U$ by Lemma 5.2(3). Hence, $U \subseteq \texttt{con}\, U$ by (2). The reverse inclusion follows from (1) since kernels are monotone. For the converse, suppose $\texttt{con}\, U = U$. By (3), $U \subseteq \operatorname{cl} W$ implies $U = \texttt{con}\, U \subseteq W$ for each $W \in \operatorname{OpUp}(X)$. Thus, U is a Scott upset by Lemma 5.2(3).

(5) Suppose that $a \ll b$ and $U \in \operatorname{OpUp}(X)$ is such that $\varphi(b) \subseteq \operatorname{cl} U$. Since $U = \bigcup \varphi[S]$ for some $S \subseteq L$ (see Lemma 2.8(2)), by (I.3), we have

$$\varphi(b) \subseteq \operatorname{cl} \bigcup \varphi[S] = \varphi\left(\bigvee S\right).$$

Therefore, $b \leq \bigvee S$. Since $a \ll b$, there is a finite $T \subseteq S$ such that $a \leq \bigvee T$. Thus,

$$\varphi(a) \subseteq \varphi\left(\bigvee T\right) = \bigcup \varphi[T] \subseteq \bigcup \varphi[S] = U.$$

Consequently, $\varphi(a) \ll \varphi(b)$.

Conversely, suppose that $\varphi(a) \ll \varphi(b)$. Therefore, $\varphi(b) \subseteq \operatorname{cl} U$ implies $\varphi(a) \subseteq U$ for each $U \in \operatorname{OpUp}(X)$. Let $b \leq \bigvee S$ for some $S \subseteq L$. Then

$$\varphi(b) \subseteq \varphi\left(\bigvee S\right) = \operatorname{cl} \bigcup \varphi[S].$$

By assumption, $\varphi(a) \subseteq \bigcup \varphi[S]$. Since $\varphi(a)$ is compact, $\varphi(a) \subseteq \varphi[T] = \varphi(\bigvee T)$ for some finite $T \subseteq S$. Thus, $a \leq \bigvee T$, and hence $a \ll b$.

This proves that $a \ll b$ iff $\varphi(a) \ll \varphi(b)$. The latter is equivalent to $\varphi(a) \subseteq \texttt{con}(b)$ by (2). $\square$

Remark 8.5. The equivalence of the first two statements of Lemma 8.4(5) was first proved in [PS88, Prop. 3.6].

The following theorem establishes that a frame is continuous precisely when `con` is representative in its Priestley space.

Theorem 8.6. *Let L be a frame and X its Priestley space. Then L is a continuous frame iff* $\mathtt{con}$ *is representative.*

Proof. Let $a \in L$. By (I.3) and Lemma 8.4(5),

$$a = \bigvee\{b \in L \mid b \ll a\} \iff \varphi(a) = \operatorname{cl} \mathtt{con}(a) \iff \mathtt{con}(a) \text{ is dense in } \varphi(a).$$

Therefore, L is continuous iff $\mathtt{con}$ is representative. □

Proposition 8.7. *Let $h\colon L_1 \to L_2$ be a frame homomorphism and $f\colon X_2 \to X_1$ its dual* L*-morphism. Then h is proper iff*

$$f^{-1}(\mathtt{con}\, U) \subseteq \mathtt{con}\, f^{-1}(U) \tag{III.1}$$

for all $U \in \mathrm{ClopUp}(X_1)$.

Proof. First, suppose that h is proper and $U \in \mathrm{ClopUp}(X_1)$. Let $x \in f^{-1}(\mathtt{con}\, U)$. Then $f(x) \in \mathtt{con}\, U$. Therefore, there exists $V \in \mathrm{ClopUp}(X_1)$ with $f(x) \in V \ll U$. Since $U, V \in \mathrm{ClopUp}(X_1)$, there exist $a, b \in L_1$ with $\varphi(a) = V$ and $\varphi(b) = U$. Then $a \ll b$ by Lemma 8.4(5). Since h is proper, $ha \ll hb$. Hence, using Lemma 8.4(5) again, $\varphi(ha) \ll \varphi(hb)$. By Remark 2.5, $f^{-1}(V) = \varphi(ha)$ and $f^{-1}(U) = \varphi(hb)$. Thus, $x \in f^{-1}(V) \ll f^{-1}(U)$, and so $x \in \mathtt{con}\, f^{-1}(U)$.

Conversely, suppose (III.1) holds for all $U \in \mathrm{ClopUp}(X_1)$. Let $a \ll b$. Then $\varphi(a) \subseteq \mathtt{con}(b)$ by Lemma 8.4(5). Therefore, $f^{-1}(\varphi(a)) \subseteq f^{-1}(\mathtt{con}(b))$. Thus, $f^{-1}(\varphi(a)) \subseteq \mathtt{con}\, f^{-1}(\varphi(b))$ by (III.1). Consequently, $f^{-1}(\varphi(a)) \ll f^{-1}(\varphi(b))$ by Lemma 8.4(2). Hence, using Remark 2.5,

$$\varphi(ha) = f^{-1}(\varphi(a)) \ll f^{-1}(\varphi(b)) = \varphi(hb),$$

and so $ha \ll hb$ by Lemma 8.4(5), yielding that h is proper. □

It is straightforward to verify that the identity L-morphism is proper and that the composition of two proper L-morphisms is again proper. That is, the category introduced in the following definition is well defined.

Definition 8.8.

(1) An L-space is L-*continuous* if con is representative.

(2) An L-morphism $f\colon X \to Y$ is *proper* provided $f^{-1}(\mathsf{con}\, U) \subseteq \mathsf{con}\, f^{-1}(U)$ for each $U \in \mathrm{ClopUp}(X)$.

(3) Let $\mathsf{ConLPries}$ be the category of continuous L-spaces and proper L-morphisms.

The name L-continuous is justified by Theorem 8.6. We now establish the dual equivalence between the category of continuous L-spaces and the category of continuous frames.

Theorem 8.9. ConFrm *is dually equivalent to* $\mathsf{ConLPries}$.

Proof. The units $\varphi\colon L \to (\mathcal{C}lopUp \circ \mathcal{X})(L)$ and $\varepsilon\colon X \to (\mathcal{X} \circ \mathcal{C}lopUp)(X)$ of Pultr–Sichler duality (see Theorem 3.2) remain isomorphisms in ConFrm and $\mathsf{ConLPries}$. Thus, it follows from Theorem 8.6 and Proposition 8.7 that the restrictions of the functors $\mathcal{X}$ and $\mathcal{C}lopUp$ yield the desired dual equivalence. □

Next, we connect $\mathsf{ConLPries}$ with the category LKSob of locally compact sober spaces. To do so, we establish a close connection between the way-below relation $\ll$ and Scott upsets in continuous L-spaces. It was shown in [PS00, Sec. 5] that $U \ll V$ iff there is a Scott upset F such that $U \subseteq F \subseteq V$. Technically, this statement in [PS00] is formulated for L-compact sets rather than Scott upsets, but the formulations are equivalent by Remark 5.4. We include proofs both for completeness and to align with our terminology, which differs from that of [PS00].

Lemma 8.10 ([PS00, Lem. 5.3]). *Let X be a continuous* L*-space and $U, V \in \mathrm{ClopUp}(X)$. If $U \ll V$, then there exists $W \in \mathrm{ClopUp}(X)$ such that $U \ll W \ll V$.*

Proof. Suppose $U \ll V$. Since X is L-continuous, $\mathtt{con}$ is representative, hence

$$\begin{aligned} V &= \mathrm{cl} \bigcup \{W \mid W \ll V\} \\ &= \mathrm{cl} \bigcup \{\mathrm{cl}\, \mathtt{con}\, W \mid W \subseteq \mathtt{con}\, V\} && \text{by Lemma 8.4(2)} \\ &= \mathrm{cl} \bigcup \{\mathtt{con}\, W \mid W \subseteq \mathtt{con}\, V\}. \end{aligned}$$

Therefore, $U \subseteq \bigcup \{\mathtt{con}\, W \mid W \subseteq \mathtt{con}\, V\}$. Since U is compact, $U \subseteq \mathtt{con}\, W_1 \cup \cdots \cup \mathtt{con}\, W_n$ for some $W_1, \dots, W_n \subseteq \mathtt{con}\, V$. Let $W = W_1 \cup \cdots \cup W_n$. Then $W \in \mathrm{ClopUp}(X)$ and $W \subseteq \mathtt{con}\, V$, so $W \ll V$ by Lemma 8.4(2). Also, $\mathtt{con}\, W_i \subseteq \mathtt{con}\, W$ because $W_i \subseteq W$ for all $i \le n$ ($\mathtt{con}$ is monotone as it is a kernel). Thus, $U \subseteq \mathtt{con}\, W_1 \cup \cdots \cup \mathtt{con}\, W_n \subseteq \mathtt{con}\, W$, and so $U \ll W$ again by Lemma 8.4(2). □

Remark 8.11. It is well known (see, e.g., [Joh82, p. 289]) that the way-below relation on a continuous frame L is interpolating, meaning that $a \ll b$ implies $a \ll c \ll b$ for some $c \in L$. Lemma 8.10 provides an alternate proof of this result in the language of Priestley spaces.

The next lemma generalizes [PS00, Lem. 4.5]. It provides a method to construct Scott upsets from suitably structured families of clopen upsets in continuous L-spaces.

Lemma 8.12. *Let X be a continuous* L*-space. If $\mathcal{U} \subseteq \mathrm{ClopUp}(X)$ is a down-directed family such that $\bigcap \mathcal{U} = \bigcap \{\mathtt{con}\, U \mid U \in \mathcal{U}\}$, then $\bigcap \mathcal{U}$ is a Scott upset.*

Proof. Clearly $\bigcap \mathcal{U} \in \mathrm{ClUp}(X)$. To see that it is a Scott upset, by Lemma 5.2 it is enough to show that $\bigcap \mathcal{U} \subseteq \mathrm{cl}\, V$ implies $\bigcap \mathcal{U} \subseteq V$ for every $V \in \mathrm{OpUp}(X)$. By (I.2), $\mathrm{cl}\, V$ is open.

Therefore, since X is compact and $\bigcap \mathcal{U}$ is down directed, from $\bigcap \mathcal{U} \subseteq \operatorname{cl} V$ it follows that there is $U \in \mathcal{U}$ with $U \subseteq \operatorname{cl} V$. Thus, $\mathtt{con}\, U \subseteq V$ by Lemma 8.4(3). Since $U \in \mathcal{U}$ and $\bigcap \mathcal{U} = \bigcap \{\mathtt{con}\, U \mid U \in \mathcal{U}\}$, we have $\bigcap \mathcal{U} \subseteq \mathtt{con}\, U$. Consequently, $\bigcap \mathcal{U} \subseteq V$. □

We now formalize the connection between the way-below relation and Scott upsets in continuous L-spaces.

Proposition 8.13. *Let X be an* L*-space and $U, V \in \operatorname{ClopUp}(X)$.*

(1) *If there is a Scott upset F with $U \subseteq F \subseteq V$, then $U \ll V$.*

(2) *If X is* L*-continuous, then the converse of* (1) *also holds.*

Proof. (1) Suppose $U \subseteq F \subseteq V$ for some $F \in \operatorname{SUp}(X)$, and let $W \in \operatorname{OpUp}(X)$ be such that $V \subseteq \operatorname{cl} W$. Then $U \subseteq F \subseteq W$ since F is a Scott-upset. Thus, $U \ll V$.

(2) Suppose $U \ll V$, so $U \subseteq \mathtt{con}\, V$ by Lemma 8.4(2). Use Lemma 8.10 to construct a sequence $\{W_n\}$ of clopen upsets such that

$$U \ll W_{n+1} \ll W_n \ll V$$

for every $n \in \mathbb{N}$. Note that $\{W_n\}$ is down directed and $\bigcap W_n = \bigcap \mathtt{con}\, W_n$. Hence, $F \coloneqq \bigcap W_n$ is a Scott-upset by Lemma 8.12. Moreover, $U \subseteq F \subseteq V$, as required. □

Assuming the Prime Ideal Theorem, every continuous frame is spatial (see, e.g., [Joh82, p. 311]). In [PS00, Prop. 4.6], an alternate proof of this fact is given using Priestley spaces:

Proposition 8.14. *If X is a continuous* L*-space, then X is* L*-spatial.*

Proof. To see that X is L-spatial, we need to show that $\operatorname{loc} X$ is dense in X. By Lemma 2.8(1), it is enough to show that $(U \setminus V) \cap \operatorname{loc} X \neq \varnothing$ for all $U, V \in \operatorname{ClopUp}(X)$ with $U \setminus V \neq \varnothing$. Let

$x \in U \setminus V$. Then $x \in \mathrm{cl}\,\mathtt{con}\,U$ since $\mathtt{con}$ is representative. Therefore, $(U \setminus V) \cap \mathtt{con}\,U \neq \varnothing$, so there is a clopen upset $W \ll U$ with $(U \setminus V) \cap W \neq \varnothing$. By Proposition 8.13(2), there is a Scott-upset F such that $W \subseteq F \subseteq U$. Thus, $(F \cap U) \setminus V \neq \varnothing$, so there is $z \in (F \cap U) \setminus V$. Since F is a Scott-upset, by Lemma 5.2(2) there is $y \in F \cap \mathrm{loc}\,X$ with $y \leq z$. But then $y \in U \setminus V$ since $F \subseteq U$ and $X \setminus V$ is a downset. Hence, $(U \setminus V) \cap \mathrm{loc}\,X \neq \varnothing$, proving that $\mathrm{loc}\,X$ is dense in X. □

As a consequence, if X is a continuous L-space, then $\mathrm{loc}\,X$ is dense in X. We now establish that a spatial L-space is L-continuous iff its localic part is locally compact. This result relies on the Priestley analogue of the Hofmann–Mislove Theorem established in the previous section (see Theorem 6.7).

Theorem 8.15. *For a spatial frame L and its Priestley space X, the following conditions are equivalent:*

(1) *L is continuous.*

(2) *X is* L-*continuous.*

(3) $\mathrm{loc}\,X$ *is locally compact.*

Proof. (1)⇔(2) This is given by Theorem 8.6.

(2)⇒(3) Suppose that X is L-continuous, $y \in \mathrm{loc}\,X$, and $\zeta(a)$ is an open neighborhood of y. Since $\zeta(a) = \varphi(a) \cap \mathrm{loc}\,X$ (see Lemma 4.9(1)), we have

$$y \in \varphi(a) = \mathrm{cl}\,\mathtt{con}(a) = \mathrm{cl} \bigcup \{\varphi(b) \mid \varphi(b) \ll \varphi(a)\}.$$

Because ${\downarrow}y$ is open, $y \in \varphi(b)$ for some $\varphi(b) \ll \varphi(a)$. Therefore, $y \in \varphi(b) \cap \mathrm{loc}\,X = \zeta(b)$. By

Proposition 8.13(2), there is a Scott upset F such that $\varphi(b) \subseteq F \subseteq \varphi(a)$. Thus,

$$y \in \zeta(b) \subseteq F \cap \operatorname{loc} X \subseteq \zeta(a).$$

By Theorem 6.7, $F \cap \operatorname{loc} X$ is compact. Consequently, $\operatorname{loc} X$ is locally compact.

(3)⇒(2) Suppose that $\operatorname{loc} X$ is locally compact and $a \in L$. We must show that $\mathtt{con}(a)$ is dense in $\varphi(a)$. Let $x \in \varphi(a)$ and W be an open neighborhood of x in X. By Lemma 2.8(1), there exist $U, V \in \operatorname{ClopUp}(X)$ such that $x \in U \setminus V \subseteq W$. Therefore, $(U \setminus V) \cap \varphi(a) \neq \varnothing$. Because L is spatial, $\operatorname{loc} X$ is dense in X (see Theorem 4.4), so $(U \setminus V) \cap \zeta(a) \neq \varnothing$, and hence there is $y \in (U \setminus V) \cap \zeta(a)$. Since $\operatorname{loc} X$ is locally compact, there is $b \in L$ and a compact saturated $K \subseteq \operatorname{loc} X$ such that $y \in \zeta(b) \subseteq K \subseteq \zeta(a)$. By Theorem 6.7, ${\uparrow}K$ is a Scott upset. Thus, ${\uparrow}K$ is closed, and so $\varphi(b) = \operatorname{cl} \zeta(b) \subseteq {\uparrow}K$ by Remark 4.10(2). Therefore, $\varphi(b) \subseteq {\uparrow}K \subseteq \varphi(a)$. Then $\varphi(b) \ll \varphi(a)$ by Proposition 8.13(1). Thus, $y \in \mathtt{con}(a)$ by Lemma 8.4(2). This implies that $(U \setminus V) \cap \mathtt{con}(a) \neq \varnothing$, so $\mathtt{con}(a)$ is dense in $\varphi(a)$. □

Theorem 8.15 establishes a one-to-one correspondence between continuous frames, continuous L-spaces, and locally compact sober spaces. We now extend this correspondence to the corresponding categorical equivalences by relating proper L-morphisms and proper continuous maps.

The following result shows that in continuous L-spaces, $\mathtt{con}$ can be described in terms of the localic part. This observation plays a key role in Proposition 8.17, which provides conditions for L-morphisms to be proper in the setting of continuous L-spaces.

Lemma 8.16. *Let X be a continuous* L*-space. Then* $\mathtt{con}\, U = {\uparrow}(U \cap \operatorname{loc} X)$ *for every* $U \in \operatorname{ClopUp}(X)$.

Proof. First, suppose that $x \in \mathtt{con}\, U$. Then there is $V \in \mathrm{ClopUp}(X)$ with $x \in V \ll U$. By Proposition 8.13(2), there is a Scott upset F with $V \subseteq F \subseteq U$. Therefore, there is $y \in F \cap \mathrm{loc}\, X$ with $y \leq x$. Thus, $x \in {\uparrow}(U \cap \mathrm{loc}\, X)$.

Conversely, suppose that $x \in {\uparrow}(U \cap \mathrm{loc}\, X)$. Then there is $y \in U \cap \mathrm{loc}\, X$ with $y \leq x$. Since $\mathtt{con}$ is representative, $U = \mathrm{cl}\, \mathtt{con}\, U$, so ${\uparrow}y \subseteq \mathrm{cl}\, \mathtt{con}\, U$. Thus, since ${\uparrow}y$ is a Scott upset and $\mathtt{con}\, U$ is an open upset, $x \in {\uparrow}y \subseteq \mathtt{con}\, U$ by Lemma 5.2(3). ☐

Proposition 8.17. *For an* L*-morphism $f \colon X \to Y$ between continuous* L*-spaces, the following conditions are equivalent:*

(1) *f is proper.*

(2) $f^{-1}{\uparrow}(U \cap \mathrm{loc}\, Y) = {\uparrow}(f^{-1}(U) \cap \mathrm{loc}\, X)$ *for all* $U \in \mathrm{ClopUp}(Y)$.

(3) $f^{-1}({\uparrow}y)$ *is a Scott upset of X for all* $y \in \mathrm{loc}\, Y$.

(4) $f^{-1}(F)$ *is a Scott upset of X for all Scott upsets F of Y.*

(5) ${\downarrow}f(x) \cap \mathrm{loc}\, Y \subseteq {\downarrow}f({\downarrow}x \cap \mathrm{loc}\, X)$ *for all* $x \in X$.

Proof. (1)⇒(2) Suppose $x \in f^{-1}{\uparrow}(U \cap \mathrm{loc}\, Y)$. Then $x \in f^{-1}(\mathtt{con}\, U)$ by Lemma 8.16. Since f is proper, $x \in \mathtt{con}\, f^{-1}(U)$, and using Lemma 8.16 again yields $x \in {\uparrow}(f^{-1}(U) \cap \mathrm{loc}\, X)$. For the reverse inclusion, suppose $x \in {\uparrow}(f^{-1}(U) \cap \mathrm{loc}\, X)$. Then $x \geq y$ for some $y \in f^{-1}(U) \cap \mathrm{loc}\, X$. Therefore, $f(x) \geq f(y)$ and $f(y) \in U$. By Lemma 4.11(1), $f(\mathrm{loc}\, X) \subseteq \mathrm{loc}\, Y$. Thus, $f(y) \in U \cap \mathrm{loc}\, Y$, so $f(x) \in {\uparrow}(Y \cap \mathrm{loc}\, Y)$, and hence $x \in f^{-1}{\uparrow}(Y \cap \mathrm{loc}\, Y)$.

(2)⇒(3) Since ${\uparrow}y$ is a closed upset, ${\uparrow}y = \bigcap\{U \in \mathrm{ClopUp}(Y) \mid y \in U\}$ by Lemma 2.8(3). Therefore, since $y \in \mathrm{loc}\, Y$, we have ${\uparrow}y = \bigcap\{{\uparrow}(U \cap \mathrm{loc}\, Y) \mid y \in U \in \mathrm{ClopUp}(Y)\}$. Thus, by

(2) and Lemma 8.16,

$$\begin{aligned}\bigcap\{f^{-1}(U) \mid y \in U \in \mathrm{ClopUp}(Y)\} &= f^{-1}\left(\bigcap\{U \in \mathrm{ClopUp}(Y) \mid y \in U\}\right) \\ &= f^{-1}\left(\bigcap\{\uparrow(U \cap \operatorname{loc} Y) \mid y \in U \in \mathrm{ClopUp}(Y)\}\right) \\ &= \bigcap\{f^{-1}\uparrow(U \cap \operatorname{loc} Y) \mid y \in U \in \mathrm{ClopUp}(Y)\} \\ &= \bigcap\{\uparrow(f^{-1}(U) \cap \operatorname{loc} X) \mid y \in U \in \mathrm{ClopUp}(Y)\} \\ &= \bigcap\{\mathtt{con}\, f^{-1}(U) \mid y \in U \in \mathrm{ClopUp}(Y)\}.\end{aligned}$$

Consequently,

$$f^{-1}(\uparrow y) = \bigcap\{f^{-1}(U) \mid y \in U \in \mathrm{ClopUp}(Y)\} = \bigcap\{\mathtt{con}\, f^{-1}(U) \mid y \in U \in \mathrm{ClopUp}(Y)\}$$

is a Scott upset by Lemma 8.12.

(3)⇒(4) Let F be a Scott upset of Y. By (3),

$$\begin{aligned}\min f^{-1}(F) &= \min f^{-1} \bigcup\{\uparrow y \mid y \in \min F\} \\ &= \min \bigcup\{f^{-1}(\uparrow y) \mid y \in \min F\} \\ &\subseteq \bigcup\{\min f^{-1}(\uparrow y) \mid y \in \min F\} \subseteq \operatorname{loc} X.\end{aligned}$$

Thus, $f^{-1}(F)$ is a Scott upset of X.

(4)⇒(5) Suppose $y_2 \in {\downarrow} f(x) \cap \operatorname{loc} Y$. Then $\uparrow y_2$ is a Scott upset of Y, so $f^{-1}(\uparrow y_2)$ is a Scott upset of X by (4). Since $x \in f^{-1}(\uparrow y_2)$, there is $y_1 \in \min f^{-1}(\uparrow y_2)$ such that $y_1 \leq x$. Therefore, $y_2 \leq f(y_1)$ and $y_1 \in {\downarrow} x \cap \operatorname{loc} X$. Thus, $y_2 \in {\downarrow} f({\downarrow} x \cap \operatorname{loc} X)$.

(5)⇒(1) Let $x \in f^{-1}(\mathtt{con}\, U)$. Then $f(x) \in \mathtt{con}\, U$, and Lemma 8.16 implies that $f(x) \in \uparrow(U \cap \operatorname{loc} Y)$. Therefore, there is $y \in {\downarrow} f(x) \cap (U \cap \operatorname{loc} Y)$. By (5), $y \in {\downarrow} f({\downarrow} x \cap \operatorname{loc} X)$, so there is $y' \in {\downarrow} x \cap \operatorname{loc} X$ with $y \leq f(y')$. Thus, $f(y') \in U$, and hence $y' \in f^{-1}(U) \cap \operatorname{loc} X$.

Consequently, Lemma 8.16 yields that

$$x \in {\uparrow}(f^{-1}(U) \cap \operatorname{loc} X) = \mathtt{con}(f^{-1}(U) \cap \operatorname{loc} X) \subseteq \mathtt{con}\, f^{-1}(U). \qquad \square$$

The proposition shows that proper L-morphisms between continuous L-spaces are precisely those that pull Scott upsets back to Scott upsets. In view of Theorem 6.7, this is the Priestley analogue of the fact that proper continuous maps between locally compact sober spaces are precisely those that pull compact saturated sets back to compact saturated sets (see Remark 7.6(1)).

We now prove that proper frame homomorphisms between frames, proper L-morphisms between their Priestley spaces, and proper continuous maps between their localic parts are in a one-to-one correspondence.

Theorem 8.18. *Let $h\colon L_1 \to L_2$ be a frame homomorphism between continuous frames, $f\colon X_2 \to X_1$ its dual* L*-morphism, and $g = \mathcal{L}oc(f)\colon \operatorname{loc} X_2 \to \operatorname{loc}(X_1)$ the restriction of f. The following are equivalent:*

1. *$h\colon L_1 \to L_2$ is a proper frame homomorphism.*
2. *$f\colon X_2 \to X_1$ is a proper* L*-morphism.*
3. *$g\colon \operatorname{loc} X_2 \to \operatorname{loc} X_1$ is a proper continuous map.*

Proof. (1)⇔(2) This follows from Proposition 8.7.

(2)⇒(3) We verify that g satisfies Definition 7.4(2). By Remark 7.6(1), it is sufficient to show that $g^{-1}(U)$ is compact for each compact saturated U in $\operatorname{loc} X_1$. Since U is compact saturated in $\operatorname{loc} X_1$, we have that ${\uparrow}U$ is a Scott upset of X_1 by Theorem 6.7. Hence, $f^{-1}({\uparrow}U)$ is a Scott upset of X_2 by Proposition 8.17(4). Thus, $f^{-1}({\uparrow}U) \cap \operatorname{loc} X_2$ is compact saturated in $\operatorname{loc} X_2$ by Theorem 6.7. But $f^{-1}(\uparrow U) \cap \operatorname{loc} X_2 = g^{-1}(U)$ because U is saturated in $\operatorname{loc} X_2$ and g is the restriction of f to $\operatorname{loc} X_2$. Therefore, $g^{-1}(U)$ is compact.

(3)⇒(2) By Proposition 8.17(3), it is enough to show that $f^{-1}(\uparrow y)$ is a Scott upset of X_2 for each $y \in \operatorname{loc} X_1$. Since $y \in \operatorname{loc} X_1$, we have that $\uparrow y$ is a Scott upset of X_1, so $\uparrow y \cap \operatorname{loc} X_1$ is compact saturated in $\operatorname{loc} X_1$ by Theorem 6.7. Because g is proper, $g^{-1}(\uparrow y \cap \operatorname{loc} X_1)$ is compact saturated in $\operatorname{loc} X_2$. Hence, $F := \uparrow g^{-1}(\uparrow y \cap \operatorname{loc} X_1)$ is a Scott upset of X_2 by Theorem 6.7. Therefore, it suffices to show that $f^{-1}(\uparrow y) = F$.

Clearly $F \subseteq f^{-1}(\uparrow y)$. For the reverse inclusion, suppose $x \notin F$. Then there exists $D \in \operatorname{ClopDn}(X_2)$ such that $x \in D$ and $D \cap g^{-1}(\uparrow y \cap \operatorname{loc} X_1) = \varnothing$ (see Lemma 2.8(3)). Hence, $y \notin \downarrow g(D \cap \operatorname{loc} X_2)$. Since g is proper, $\downarrow g(D \cap \operatorname{loc} X_2) \cap \operatorname{loc} X_1$ is closed in $\operatorname{loc} X_1$, and so $\downarrow g(D \cap \operatorname{loc} X_2) \cap \operatorname{loc} X_1 = E \cap \operatorname{loc} X_1$ for some $E \in \operatorname{ClopDn}(X_1)$. Therefore, $y \notin E$ and $g(D \cap \operatorname{loc} X_2) \subseteq E$, so $\downarrow \operatorname{cl} g(D \cap \operatorname{loc} X_2) \subseteq E$. Because X_1 and X_2 are L-continuous, they are L-spatial by Proposition 8.14. Thus,

$$\downarrow f(D) = \downarrow f \operatorname{cl}(D \cap \operatorname{loc} X_2) = \downarrow \operatorname{cl} f(D \cap \operatorname{loc} X_2) = \downarrow \operatorname{cl} g(D \cap \operatorname{loc} X_2) \subseteq E,$$

where the second equality holds because f is a closed map (hence f commutes with closure, see, e.g., [Eng89, p. 35]). Consequently, $y \notin \downarrow f(D)$, and hence $x \notin f^{-1}(\uparrow y)$. □

Corollary 8.19. *Suppose X and Y are continuous* L*-spaces, and let $g\colon \operatorname{loc} X \to \operatorname{loc} Y$ be a proper continuous map between their localic parts. Then there exists a proper* L*-morphism $f\colon X \to Y$ extending g.*

Proof. By Proposition 4.18, there is an L-morphism $f : X \to Y$ extending g. Therefore, $\mathscr{L}oc(f) = g$, and so f is proper by Theorem 8.18. □

We now prove that the categories of continuous L-spaces and locally compact sober spaces are equivalent.

Theorem 8.20. $\mathsf{ConLPries}$ *is equivalent to* LKSob*.*

Proof. By Theorem 8.15, the restriction of $\mathscr{L}oc$ is well defined on objects. By Theorem 8.18, the restriction of $\mathscr{L}oc$ is also well defined on morphisms. This together with Corollary 8.19 shows that Theorems 4.15 and 4.19 apply, yielding that $\mathscr{L}oc\colon \mathsf{ConLPries} \to \mathsf{LKSob}$ is essentially surjective, full, and faithful. □

This equivalence, together with the previously established duality between continuous frames and continuous L-spaces, provides an alternative proof of Hofmann–Lawson duality.

Corollary 8.21 (Hofmann–Lawson)**.** ConFrm *is dually equivalent to* LKSob*.*

Proof. Combine Theorems 8.9 and 8.20. □

This concludes our proof of Hofmann–Lawson duality through Priestley spaces, offering a new perspective on this fundamental result in pointfree topology via, among other things, the continuous kernel `con`.

9 Priestley spaces of stably continuous frames: charting stability

To derive the two dualities for stably continuous frames (see Theorem 7.7(2,3)), we first characterize stability of the way-below relation $\ll$ in the language of Priestley spaces. Stability strengthens continuity by requiring that $\ll$ is preserved under binary meets.

In the previous section, we described continuity of a frame in terms of the kernel `con` being representative in its Priestley space. Here, we extend this approach by showing that stability can also be characterized in terms of this kernel, specifically by how it interacts with intersections. This leads to a description of the Priestley spaces of stably continuous frames.

Lemma 9.1. *Let L be a continuous frame and X its Priestley space. For $a, b \in L$, we have $\texttt{con}(a) \cap \texttt{con}(b) = \texttt{con}(a \wedge b)$ iff $c \ll a, b$ implies $c \ll a \wedge b$ for all $c \in L$.*

Proof. First, suppose that $\texttt{con}(a) \cap \texttt{con}(b) = \texttt{con}(a \wedge b)$. Let $c \in L$ with $c \ll a, b$. Then $\varphi(c) \subseteq \texttt{con}(a), \texttt{con}(b)$ by Lemma 8.4(5). Therefore, $\varphi(c) \subseteq \texttt{con}(a) \cap \texttt{con}(b) = \texttt{con}(a \wedge b)$. Thus, $c \ll a \wedge b$ using Lemma 8.4(5) again.

For the converse, suppose that $c \ll a, b$ implies $c \ll a \wedge b$ for all $c \in L$. Then Lemma 8.4(5) gives that for all $c \in L$,

$$\varphi(c) \ll \varphi(a), \varphi(b) \text{ implies } \varphi(c) \ll \varphi(a) \cap \varphi(b). \tag{III.2}$$

Since $\texttt{con}$ is monotone (as it is a kernel), $\texttt{con}(a \wedge b) \subseteq \texttt{con}(a) \cap \texttt{con}(b)$. For the reverse inclusion, let $x \in \texttt{con}(a) \cap \texttt{con}(b)$. Then there are $d, e \in L$ with $x \in \varphi(d) \ll \varphi(a)$ and $x \in \varphi(e) \ll \varphi(b)$. Let $c = d \wedge e$. Then $x \in \varphi(c) \ll \varphi(a), \varphi(b)$. Consequently, by (III.2), $\varphi(c) \ll \varphi(a) \cap \varphi(b) = \varphi(a \wedge b)$. Therefore, $x \in \texttt{con}(a \wedge b)$ by Lemma 8.4(5). Thus, $\texttt{con}(a) \cap \texttt{con}(b) = \texttt{con}(a \wedge b)$. □

Recall (see Definition 8.1(3)) that we defined a kernel to be stable if it commutes with binary intersections. The previous lemma motivates this definition and it aligns with the idea that stability is the fact that $\ll$ is preserved under binary meets. Recall also that an L-space X is L-compact if $\min X \subseteq \operatorname{loc} X$ (see Definition 5.10).

Definition 9.2.

(1) An L-space is L-*stably continuous* if it is L-continuous and `con` is stable.

(2) An L-space is L-*stably compact* if it is L-stably continuous and L-compact.

(3) Let StCLPries be the full subcategory of ConLPries consisting of stably continuous L-spaces.

(4) Let StKLPries be the full subcategory of StCLPries consisting of stably compact L-spaces.

Remark 9.3. In [BM23], stably continuous L-spaces are defined to be continuous L-spaces X such that $\mathrm{SUp}(X)$ is closed under binary intersections. In Lemma 9.6, we will see that for continuous L-spaces, this property and the condition that `con` is stable are equivalent.

The following theorem formalizes the connection between stably continuous and stably compact frames and the L-spaces of the same name.

Theorem 9.4. *Let L be a frame and X its Priestley space.*

(1) *L is a stably continuous frame iff X is a stably continuous* L-*space.*

(2) *L is a stably compact frame iff X is a stably compact* L-*space.*

Proof. (1) Apply Theorem 8.6 and Lemma 9.1.

(2) This follows from (1) and Theorem 5.8(2). □

As a consequence, we obtain the following dual equivalences:

Corollary 9.5.

(1) StCFrm *is dually equivalent to* StCLPries.

(2) StKFrm *is dually equivalent to* StKLPries.

Proof. (1) Restrict Theorem 8.9 to the full subcategories StCFrm and $\mathsf{StCLPries}$ using Theorem 9.4(1).

(2) Similar to (1), but now use Theorem 9.4(2). □

Next, we establish the connection between stably continuous and stably compact L-spaces and their topological counterparts StLKSp and StKSp. The following lemma provides a characterization of stability in terms of Scott upsets.

Lemma 9.6. *Let X be a continuous* L*-space.*

(1) *For every Scott upset F, we have $F = \bigcap\{\mathtt{con}\, U \mid F \subseteq U \in \mathrm{ClopUp}(X)\}$.*

(2) $\mathrm{SUp}(X)$ *is closed under binary intersections iff* $\mathtt{con}$ *is stable.*

Proof. (1) Suppose $F \subseteq U \in \mathrm{ClopUp}(X)$. Since X is L-continuous, $\mathtt{con}$ is representative, and hence $F \subseteq \mathrm{cl}\, \mathtt{con}\, U$. By Lemma 8.4(1), $\mathtt{con}\, U \in \mathrm{OpUp}(X)$. Therefore, $F \subseteq \mathtt{con}\, U$ by Lemma 5.2(3). Thus, $F \subseteq \bigcap\{\mathtt{con}\, U \mid F \subseteq U \in \mathrm{ClopUp}(X)\}$. For the reverse inclusion, by Lemma 2.8(3), we have

$$F = \bigcap\{U \mid F \subseteq U \in \mathrm{ClopUp}(X)\} \supseteq \bigcap\{\mathtt{con}\, U \mid F \subseteq U \in \mathrm{ClopUp}(X)\}.$$

(2) Suppose $\mathrm{SUp}(X)$ is closed under binary intersections and $U, V \in \mathrm{ClopUp}(X)$. Since $\mathtt{con}\, U$ is an open upset for each U (see Lemma 8.4(1)), it suffices to show that

$$W \subseteq \mathtt{con}(U) \cap \mathtt{con}(V) \text{ iff } W \subseteq \mathtt{con}(U \cap V)$$

for each $W \in \mathrm{ClopUp}(X)$ (see Lemma 2.8(2)). By Lemma 8.4(2), $W \subseteq \mathtt{con}(U) \cap \mathtt{con}(V)$ iff $W \ll U$ and $W \ll V$. By Proposition 8.13, this happens iff there are $F, G \in \mathrm{SUp}(X)$ such that $W \subseteq F \subseteq U$ and $W \subseteq G \subseteq V$. By assumption, the latter is equivalent to the existence of a Scott upset H such that $W \subseteq H \subseteq U \cap V$. By invoking Proposition 8.13 again, this is equivalent to $W \ll U \cap V$, which in turn is equivalent to $W \subseteq \mathtt{con}(U \cap V)$ by Lemma 8.4(2). Thus, con is stable.

Conversely, suppose con is stable and let $F, G \in \mathrm{SUp}(X)$. If U, V, W range over clopen upsets of X, by (1) we have

$$\begin{aligned} F \cap G &= \bigcap\{\mathtt{con}\, U \mid F \subseteq U\} \cap \bigcap\{\mathtt{con}\, V \mid G \subseteq V\} \\ &= \bigcap\{\mathtt{con}\, U \cap \mathtt{con}\, V \mid F \subseteq U, G \subseteq V\} \\ &= \bigcap\{\mathtt{con}(U \cap V) \mid F \subseteq U, G \subseteq V\} \\ &= \bigcap\{\mathtt{con}\, W \mid F \cap G \subseteq W\} \\ &\subseteq \bigcap\{W \mid F \cap G \subseteq W\} = F \cap G, \end{aligned}$$

where the last equality follows from Lemma 2.8(3). For the second to last equality it is enough to observe that by compactness, $F \cap G \subseteq W$ is equivalent to $U \cap V \subseteq W$ for some clopen upsets $U \supseteq F$ and $V \supseteq G$. Thus, $F \cap G$ is a Scott upset by Lemma 8.12, completing the proof. □

The following theorem establishes a precise relationship between stably continuous L-spaces and their localic parts.

Theorem 9.7. *Let X be a spatial* L*-space.*

1. *X is a stably continuous* L*-space iff* $\mathrm{loc}\, X$ *is stably locally compact.*
2. *X is a stably compact* L*-spaces iff* $\mathrm{loc}\, X$ *is stably compact.*

Proof. (1) Suppose X is L-stably continuous. By Theorem 8.15, $\mathrm{loc}\, X$ is locally compact. Let K and J be compact saturated in $\mathrm{loc}\, X$. Then ${\uparrow}K$ and ${\uparrow}J$ are Scott upsets by Theorem 6.7. Since con is stable, ${\uparrow}K \cap {\uparrow}J$ is a Scott upset by Lemma 9.6(2). Therefore,

$K \cap J = {\uparrow}K \cap {\uparrow}J \cap \operatorname{loc} X$ is compact saturated by Theorem 6.7.

Conversely, suppose that $\operatorname{loc} X$ is stably locally compact. By Theorem 8.15, X is L-continuous. Let $U, V \in \operatorname{ClopUp}(X)$. Since $\mathtt{con}$ is monotone (as it is a kernel), it suffices to show that $\mathtt{con}\, U \cap \mathtt{con}\, V \subseteq \mathtt{con}(U \cap V)$. Let $x \in \mathtt{con}\, U \cap \mathtt{con}\, V$. Then there exist $U', V' \in \operatorname{ClopUp}(X)$ containing x such that $U' \ll U$ and $V' \ll V$. By Proposition 8.13(2) (X is L-continuous), there are Scott upsets F, G with $U' \subseteq F \subseteq U$ and $V' \subseteq G \subseteq V$. By Theorem 6.7, $F \cap \operatorname{loc} X$ and $G \cap \operatorname{loc} X$ are compact saturated. Since $\operatorname{loc} X$ is stably locally compact, $F \cap G \cap \operatorname{loc} X$ is compact saturated. Hence, ${\uparrow}(F \cap G \cap \operatorname{loc} X)$ is a Scott upset by Theorem 6.7. Moreover, because $F \cap G \subseteq U \cap V$, we have

$$
{\uparrow}(F \cap G \cap \operatorname{loc} X) \subseteq {\uparrow}(U \cap V \cap \operatorname{loc} X) = \mathtt{con}(U \cap V)
$$

by Lemma 8.16. Therefore, since X is L-spatial,

$$
x \in U' \cap V' = \operatorname{cl}(U' \cap V' \cap \operatorname{loc} X) \subseteq {\uparrow}(F \cap G \cap \operatorname{loc} X) \subseteq \mathtt{con}(U \cap V).
$$

(2) This follows from (1) by applying Lemma 5.12. $\square$

These results yield the following equivalences:

Corollary 9.8.

(1) $\mathsf{StCLPries}$ *is equivalent to* StLKSp.

(2) $\mathsf{StKLPries}$ *is equivalent to* StKSp.

Proof. (1) Restrict Theorem 8.20 to the full subcategories $\mathsf{StCLPries}$ and StLKSp using Theorem 9.7(1).

(2) This follows from (1) using Theorem 9.7(2). $\square$

As a consequence of Corollaries 9.5 and 9.8, we obtain the following well-known dualities for stably continuous frames (see Theorem 7.7(2,3)):

Corollary 9.9.

(1) StCFrm *is dually equivalent to* StLKSp*.*

(2) StKFrm *is dually equivalent to* StKSp*.*

This completes our charting of stability in Priestley spaces, showing that stably continuous and stably compact frames correspond precisely to continuous L-spaces where the kernel con is stable.

10 Priestley spaces of compact regular frames: the path to Isbell

In this section, we provide a new proof of Isbell duality between the categories KRFrm of compact regular frames and KHaus of compact Hausdorff spaces using Priestley duality. The Priestley spaces of compact regular frames were previously described in [BGJ16]. Here, we extend this description by establishing the equivalence between these L-spaces and compact Hausdorff spaces.

A fundamental ingredient in this correspondence is the well-inside relation $\prec$, which provides the formulation of regularity in frames (see Definition 7.1(2)). A characterization of this relation in terms of Priestley spaces was given in [BGJ16, Sec. 3]. Similar to how the kernel con was introduced to characterize continuity (see Definition 8.2), the well-inside relation induces the notion of regular part, which plays an analogous role in describing regularity in the Priestley setting.

Definition 10.1. Let X be an L-space.

(1) For $U, V \in \mathrm{ClopUp}(X)$, we write $U \prec V$ if ${\downarrow}U \subseteq V$.

(2) The *regular part* of U is defined by

$$\texttt{reg}\, U = \bigcup \{V \in \mathrm{ClopUp}(X) \mid V \prec U\}.$$

When $U = \varphi(a)$, we write $\texttt{reg}(a)$ for $\texttt{reg}\, U$.

This definition is motivated by the following result, which expresses the regular part in terms of upsets and downsets.

Lemma 10.2 ([BGJ16, Sec. 3])**.** *Let X be an* L*-space. For each $U \in \mathrm{ClopUp}(X)$ we have*

$$\texttt{reg}\, U = X \setminus {\downarrow}{\uparrow}(X \setminus U).$$

In particular, if X is the Priestley space of a frame L, then for $a, b \in L$,

$$a \prec b \text{ iff } \varphi(a) \prec \varphi(b) \text{ iff } \varphi(a) \subseteq \texttt{reg}(b).$$

The next result shows that `reg` naturally fits within our language of kernels, reinforcing the rigidity of this framework. Unlike `con`, the regular part is always stable, which aligns with the fact that the well-inside relation is preserved under binary intersections.

Lemma 10.3. *Let X be an* L*-space. Then* $\texttt{reg}\colon \mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$ *is a stable kernel.*

Proof. That `reg` is a kernel follows immediately from the definition. To see that it is stable, let $U, V \in \mathrm{ClopUp}(X)$. By Lemma 2.8(2) it suffices to show that

$$W \subseteq \texttt{reg}\, U \cap \texttt{reg}\, V \text{ iff } W \subseteq \texttt{reg}(U \cap V)$$

for each $W \in \mathrm{ClopUp}(X)$. We have

$$\begin{aligned} W \subseteq \mathtt{reg}\, U \cap \mathtt{reg}\, V &\iff W \subseteq \mathtt{reg}\, U \text{ and } W \subseteq \mathtt{reg}\, V \\ &\iff {\downarrow} W \subseteq U \text{ and } {\downarrow} W \subseteq V \\ &\iff {\downarrow} W \subseteq U \cap V \\ &\iff W \subseteq \mathtt{reg}(U \cap V). \end{aligned}$$

□

In our terminology, a frame L is regular iff `reg` is representative in the Priestley space of L (see [BGJ16, Lem. 3.6]). We will now establish several consequences of `reg` being representative, which in the L-spatial case characterize `reg` being representative. To do so, we first prove the following result.

Lemma 10.4. *Let X be an* L*-space, $x \in X$, $Z \subseteq X$, and $U \in \mathrm{ClopUp}(X)$.*

(1) *$x \in \mathtt{reg}\, U$ iff ${\downarrow}{\uparrow} x \subseteq U$.*

(2) *$Z \subseteq \mathtt{reg}\, U$ iff ${\downarrow}{\uparrow} Z \subseteq U$.*

Proof. (1) By Lemma 10.2,

$$\begin{aligned} x \in \mathtt{reg}\, U &\iff x \notin {\downarrow}{\uparrow}(X \setminus U) \\ &\iff {\uparrow} x \cap {\uparrow}(X \setminus U) = \varnothing \\ &\iff {\downarrow}{\uparrow} x \cap (X \setminus U) = \varnothing \\ &\iff {\downarrow}{\uparrow} x \subseteq U. \end{aligned}$$

(2) This follows from (1) since ${\downarrow}{\uparrow} Z = \bigcup \{ {\downarrow}{\uparrow} x \mid x \in Z \}$. □

Proposition 10.5. *Let X be an* L*-space and $U \in \mathrm{ClopUp}(X)$. The following three conditions are equivalent:*

(1) $U \cap \mathrm{loc}\, X \subseteq \mathtt{reg}\, U$.

(2) *For each $y \in U \cap \mathrm{loc}\, X$ there are disjoint $V, W \in \mathrm{ClopUp}(X)$ such that $y \in V$ and $X \setminus U \subseteq W$.*

(3) $\downarrow\uparrow(U \cap \mathrm{loc}\, X) \subseteq U$.

Moreover, the following condition implies conditions (1)–(3)*.*

(4) $\mathtt{reg}\, U$ *is dense in* U*.*

Furthermore, if X is L*-spatial, all four conditions are equivalent.*

Proof. (1)⇒(2) Let $y \in U \cap \mathrm{loc}\, X$. By (1), $y \in \mathtt{reg}\, U$. Hence, there exists $V \in \mathrm{ClopUp}(X)$ such that $y \in V$ and $\downarrow V \subseteq U$. By Lemma 3.4(1), $\downarrow V \in \mathrm{ClopDn}(X)$. Therefore, $W := X \setminus \downarrow V$ is a clopen upset disjoint from V such that $X \setminus U \subseteq W$.

(2)⇒(3) Let $x \in \downarrow\uparrow(U \cap \mathrm{loc}\, X)$. Then there is $y \in U \cap \mathrm{loc}\, X$ such that $x \in \downarrow\uparrow y$. By (2), there are disjoint $V, W \in \mathrm{ClopUp}(X)$ such that $y \in V$ and $X \setminus U \subseteq W$. Thus,

$$x \in \downarrow\uparrow y \subseteq \downarrow V \subseteq X \setminus W \subseteq U.$$

(3)⇒(1) Apply Lemma 10.4(2).

Therefore, conditions (1)–(3) are equivalent.

(4)⇒(1) Let $y \in U \cap \mathrm{loc}\, X$. Then $y \in \mathrm{cl}\, \mathtt{reg}\, U$ by (4). Since $\downarrow y$ is open and $\mathtt{reg}\, U$ is an upset, we conclude that $y \in \mathtt{reg}\, U$.

Let X be L-spatial.

(1)⇒(4) From $U \cap \mathrm{loc}\, X \subseteq \mathtt{reg}\, U$ it follows that $\mathrm{cl}(U \cap \mathrm{loc}\, X) \subseteq \mathrm{cl}\, \mathtt{reg}\, U$. Since X is L-spatial, $\mathrm{cl}(U \cap \mathrm{loc}\, X) = U$ (see Theorem 4.4). Thus, $\mathtt{reg}\, U$ is dense in U. □

Remark 10.6. Proposition 10.5 provides a set of conditions that characterize regularity in terms of L-spaces. The second condition is especially reminiscent of the usual definition of regularity in topological spaces. However, while the first three conditions are generally equivalent, the final condition, that $\texttt{reg}\,U$ is dense in U, is in general stronger. This means that $\texttt{reg}$ being representative (which is equivalent to the corresponding frame being regular) is stronger than the first three conditions holding for arbitrary clopen upsets. Nonetheless, in the spatial case the conditions do become equivalent to regularity.

Since regularity is captured in Priestley spaces by the kernel $\texttt{reg}$, it is natural to define regular L-spaces in terms of $\texttt{reg}$ being representative.

Definition 10.7.

(1) An L-space X is L-*regular* if $\texttt{reg}$ is representative.

(2) Let $\mathsf{KRLPries}$ be the full subcategory of $\mathsf{KLPries}$ consisting of compact regular L-spaces.

The following theorem, originally established in [BGJ16, Sec. 3] (see also [PS88, Sec. 3]), formalizes the connection between regularity in frames and L-spaces.

Theorem 10.8. *Let L be a frame and X its Priestley space.*

(1) ([BGJ16, Lem. 3.6]) *L is regular iff X is* L-*regular.*

(2) ([BGJ16, Thm. 3.9]) *L is compact regular iff X is* L-*compact and* L-*regular.*

As a consequence, we obtain the following dual equivalence.

Corollary 10.9. KRFrm *is dually equivalent to* $\mathsf{KRLPries}$*.*

Proof. Apply Theorems 3.2 and 10.8(2). □

Next, we establish the connection between compact regular L-spaces and compact Hausdorff spaces. The following result provides an analogue of the well-known fact that every compact regular frame is spatial (see, e.g., [Joh82, p. 30]).

Lemma 10.10. *If X is a compact regular* L*-space, then X is* L*-spatial.*

Proof. By Lemma 2.8(1), it is enough to show that $U \setminus V \neq \varnothing$ implies $(U \setminus V) \cap \operatorname{loc} X \neq \varnothing$ for all $U, V \in \operatorname{ClopUp}(X)$. Since X is L-regular, $U = \operatorname{cl} \mathtt{reg}\, U$. Therefore, $U \setminus V \neq \varnothing$ implies $\mathtt{reg}\, U \setminus V \neq \varnothing$. Let $z \in \mathtt{reg}\, U \setminus V$. Because $z \in \mathtt{reg}\, U$, there exists $W \in \operatorname{ClopUp}(X)$ containing z such that ${\downarrow} W \subseteq U$. By Lemma 2.9(2), there is $y \in \min({\downarrow} W)$ with $y \leq z$. Consequently, $y \in {\downarrow} W \subseteq U$ and $y \in X \setminus V$ since $X \setminus V$ is a downset. Moreover, $y \in \operatorname{loc} X$ because $y \in \min X$ and $\min X \subseteq \operatorname{loc} X$ since X is L-compact. □

The next theorem relates regularity and compactness in L-spaces to their localic parts.

Theorem 10.11. *Let X be an* L*-space.*

(1) *If X is* L*-regular, then* $\operatorname{loc} X$ *is regular.*

(2) *If X is* L*-compact and* L*-regular, then* $\operatorname{loc} X$ *is compact regular.*

If X is L*-spatial, then the converses of* (1) *and* (2) *also hold.*

Proof. (1) First, suppose X is L-regular. Let $y \in \operatorname{loc} X$ and F be a closed subset of $\operatorname{loc} X$ with $y \notin F$. Then $\operatorname{loc} X \setminus F$ is an open subset of $\operatorname{loc} X$ containing y. Therefore, there exists $U \in \operatorname{ClopUp}(X)$ with $U \cap \operatorname{loc} X = \operatorname{loc} X \setminus F$. Since X is L-regular, $\mathtt{reg}\, U$ is dense in U. Thus, by the implication (4)⇒(2) in Proposition 10.5, there exist disjoint $V, W \in \operatorname{ClopUp}(X)$ such that $y \in V$ and $X \setminus U \subseteq W$. Hence, $V \cap \operatorname{loc} X$ and $W \cap \operatorname{loc} X$ are disjoint open subsets of $\operatorname{loc} X$ such that $y \in V \cap \operatorname{loc} X$ and $F = \operatorname{loc} X \setminus U \subseteq W \cap \operatorname{loc} X$. This implies that $\operatorname{loc} X$ is regular.

For the converse, suppose X is L-spatial and $\operatorname{loc} X$ is regular. To see that $\mathtt{reg}$ is representative, let $U \in \mathrm{ClopUp}(X)$ and $y \in U \cap \operatorname{loc} X$. Since $\operatorname{loc} X$ is regular, $\operatorname{loc} X \setminus U$ is closed in $\operatorname{loc} X$, and $y \notin \operatorname{loc} X \setminus U$, there exist $V, W \in \mathrm{ClopUp}(X)$ such that $V \cap W \cap \operatorname{loc} X = \varnothing$, $y \in V$, and $\operatorname{loc} X \setminus U \subseteq W \cap \operatorname{loc} X$. Therefore, since X is L-spatial,

$$V \cap W = \mathrm{cl}(V \cap \operatorname{loc} X) \cap \mathrm{cl}(W \cap \operatorname{loc} X) = \mathrm{cl}(V \cap W \cap \operatorname{loc} X) = \mathrm{cl}\, \varnothing = \varnothing,$$

where the second equality follows from Lemma 4.16(3). Moreover, $\operatorname{loc} X \setminus U \subseteq W \cap \operatorname{loc} X$ implies that $X \setminus U \subseteq W$ because X is L-spatial. Thus, $\mathtt{reg}\, U$ is dense in U by Proposition 10.5. Consequently, $\mathtt{reg}$ is representative.

(2) This follows from (1) and Lemma 5.12. The converse follows from the converse of (1) and Lemma 5.12. □

As a consequence, we obtain the following equivalence.

Corollary 10.12. KRLPries *is equivalent to* KHaus.

Proof. Apply Corollary 4.20, Lemma 10.10, and Theorem 10.11(2). □

We now derive Isbell duality as an immediate consequence of Corollaries 10.9 and 10.12.

Corollary 10.13 (Isbell duality)**.** KRFrm *is dually equivalent to* KHaus.

As discussed in the introduction of this chapter, KRFrm is a full subcategory of StKFrm. We now work towards showing that KRLPries is a full subcategory of StKLPries. To do so, we compare $\mathtt{con}$ with $\mathtt{reg}$, which serves as a parallel to the comparison between compact and complemented elements in frames.

For two kernels $\texttt{ker}_1, \texttt{ker}_2 \colon \mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$ we write $\texttt{ker}_1 \leq \texttt{ker}_2$ provided $\texttt{ker}_1 U \subseteq \texttt{ker}_2 U$ for each $U \in \mathrm{ClopUp}(X)$. We also write $\texttt{ker}_1 = \texttt{ker}_2$ if $\texttt{ker}_1 \leq \texttt{ker}_2$ and $\texttt{ker}_2 \leq \texttt{ker}_1$.

Lemma 10.14. *Let X be an* L*-space.*

(1) *X is* L*-compact iff* $\texttt{reg} \leq \texttt{con}$.

(2) *If X is* L*-regular, then* $\texttt{con} \leq \texttt{reg}$.

(3) *If X is* L*-compact and* L*-regular, then* $\texttt{reg} = \texttt{con}$.

Proof. (1) First, suppose that X is L-compact and $U \in \mathrm{ClopUp}(X)$. We show that $V \prec U$ implies $V \ll U$ for every $V \in \mathrm{ClopUp}(X)$. Let $U \subseteq \mathrm{cl}\, W$ for some $W \in \mathrm{OpUp}(X)$. Then $U \cap \mathrm{loc}\, X \subseteq W$ by Lemma 4.16(1). Moreover, since ${\downarrow} V \subseteq U$, we have $\min({\downarrow} V) \subseteq U$. Therefore, $\min({\downarrow} V) \subseteq U \cap \mathrm{loc}\, X$ because X is L-compact. Thus,

$$V \subseteq {\uparrow} \min({\downarrow} V) \subseteq {\uparrow}(U \cap \mathrm{loc}\, X) \subseteq W.$$

Consequently, $V \ll U$, and hence $\texttt{reg}\, U \subseteq \texttt{con}\, U$.

Conversely, since $\texttt{reg}\, X = X$ (see Lemma 10.2), $\texttt{reg}\, X \subseteq \texttt{con}\, X$ implies $\texttt{con}\, X = X$, so X is a Scott upset by Lemma 8.4(4). Therefore, $\min X \subseteq \mathrm{loc}\, X$, showing that X is L-compact.

(2) Since $\texttt{reg}$ is representative, $\mathrm{cl}\, \texttt{reg}\, U = U$. Therefore, $\texttt{con}\, U \subseteq \texttt{reg}\, U$ by Lemma 8.4(3).

(3) This follows from (1) and (2). □

Following [BGJ16, p. 377], we call a subset of a poset a *biset* if it is both an upset and a downset. We will see in the next section that complemented elements of a frame correspond to clopen bisets of its Priestley space. By Theorem 5.8(1), compact elements correspond to clopen Scott upsets. Thus, the comparison between complemented elements and compact elements can also be done by comparing clopen bisets and clopen Scott upsets. In this sense the next result is a generalization of the comparison since it considers arbitrary closed bisets and Scott upsets.

Lemma 10.15. *Let X be an* L*-space. If X is* L*-compact, then*

(1) *each closed biset is a Scott upset.*

If X is L*-regular, then*

(2) *each Scott upset is a biset,*

(3) $\operatorname{loc} X \subseteq \min X$.

If X is L*-compact and* L*-regular, then*

(4) *closed bisets are exactly Scott upsets,*

(5) $\min X = \operatorname{loc} X$.

Proof. (1) Since X is L-compact, $\min X \subseteq \operatorname{loc} X$. Therefore, for each closed biset F, we have $\min F \subseteq \min X \subseteq \operatorname{loc} X$. Thus, F is a Scott upset.

(2) Suppose F is a Scott upset. Let $x \in {\downarrow}F$. Then there is $z \in F$ with $x \leq z$. Since F is a Scott upset, there is $y \in F \cap \operatorname{loc} X$ with $y \leq z$. If $y \not\leq x$, then by Priestley separation (see (I.1)), there exists $U \in \operatorname{ClopUp}(X)$ with $y \in U$ and $x \notin U$. Since X is L-regular, we have ${\downarrow}{\uparrow}(U \cap \operatorname{loc} X) \subseteq U$ by the implication (4)⇒(3) in Proposition 10.5. Therefore, ${\downarrow}{\uparrow}y \subseteq U$, so $x \in U$, a contradiction. Thus, we must have $y \leq x$, so $x \in F$, and hence F is a biset.

(3) Let $y \in \operatorname{loc} X$. Then ${\uparrow}y$ is a Scott upset, so ${\uparrow}y$ is a downset by (2). Thus, $y \in \min X$.

(4) This follows from (1) and (2).

(5) Because X is L-compact, $\min X \subseteq \operatorname{loc} X$. The reverse inclusion follows from (3). □

Remark 10.16. The frame-theoretic interpretation of Lemma 10.15(5) is that in a compact regular frame, the minimal prime filters are precisely the completely prime filters. This was first observed in [BGJ16, Lem. 5.2 and 5.3].

Proposition 10.17. *Each compact regular* L*-space is* L*-stably compact.*

Proof. Let X be a compact regular L-space. Then $\mathtt{reg}$ is representative and stable (see Lemma 10.3). By Lemma 10.14(3), $\mathtt{reg} = \mathtt{con}$, so $\mathtt{con}$ is representative and stable because $\mathtt{reg}$ is. Therefore, X is stably L-continuous. Consequently, X is L-stably compact since X is L-compact. □

The next result compares how L-regularity and L-compactness interact with L-morphisms. In particular, it shows that a compact-to-regular L-morphism must be proper.

Proposition 10.18. *Let $f\colon X \to Y$ be an* L*-morphism between* L*-spaces.*

(1) $f^{-1}(\mathtt{reg}\, U) \subseteq \mathtt{reg}\, f^{-1}(U)$ *for each* $U \in \mathrm{ClopUp}(Y)$*.*

(2) *If X is* L*-compact and Y is* L*-regular, then f is proper.*

Proof. (1) Suppose $x \in f^{-1}(\mathtt{reg}\, U)$. Then $f(x) \in \mathtt{reg}\, U$. Therefore, ${\downarrow}{\uparrow} f(x) \subseteq U$ by Lemma 10.4(1). Since f is order-preserving, we obtain $f({\downarrow}{\uparrow} x) \subseteq U$. Thus, ${\downarrow}{\uparrow} x \subseteq f^{-1}(U)$, and so $x \in \mathtt{reg}\, f^{-1}(U)$ by applying Lemma 10.4(1) again.

(2) Let $U \in \mathrm{ClopUp}(Y)$. By Lemma 10.14(1), $\mathtt{reg} \le \mathtt{con}$ in X, and by Lemma 10.14(2), $\mathtt{con} \le \mathtt{reg}$ in Y. Therefore, by (1),

$$f^{-1}(\mathtt{con}\, U) \subseteq f^{-1}(\mathtt{reg}\, U) \subseteq \mathtt{reg}\, f^{-1}(U) \subseteq \mathtt{con}\, f^{-1}(U).$$

Thus, f is proper. □

Remark 10.19. Proposition 10.18(2) corresponds to the well-known fact that every frame homomorphism from a compact frame to a regular frame is proper.

Combining Propositions 10.17 and 10.18(2), we obtain the following:

Theorem 10.20. KRLPries *is a full subcategory of* StKLPries*.*

We thus arrive at Fig. 7, completing our journey from Hofmann–Lawson to Isbell.

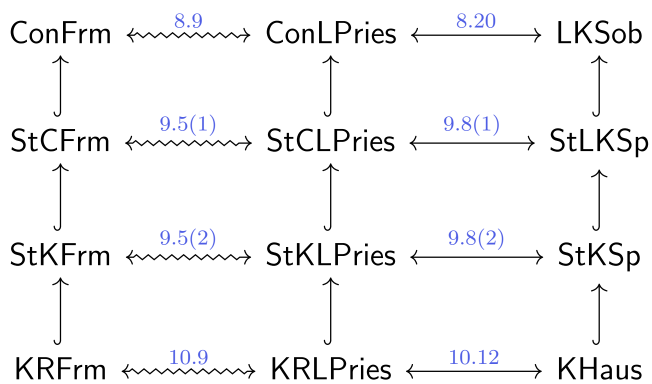

Figure 7: Equivalences and dual equivalences between various categories of continuous frames, continuous L-spaces, and locally compact sober spaces.

Chapter IV

Finding Priestley and Stone in algebraic frames

Building on the previous chapter's framework for continuous frames, this chapter develops Priestley duality for algebraic frames and demonstrates its connection to well-known dualities in pointfree topology through alternative proofs.

Recall that a complete lattice is algebraic if every element is a join of compact elements. Algebraic lattices arise naturally in various contexts. For example, the lattice of congruences of any algebra is algebraic, and up to isomorphism, every algebraic lattice arises in this way (see, e.g., [BS81]). By a well-known result of Nachbin [Nac49] (see also [BF48]), algebraic lattices are precisely the ideal lattices of join-semilattices. By [Grä11, Lem. 184], the ideal lattice of a join-semilattice is distributive iff the join-semilattice is distributive. Since algebraic frames are precisely the distributive algebraic lattices (see, e.g., [Grä11, p. 165]), it follows that they correspond exactly to the ideal lattices of distributive join-semilattices.

Algebraic frames have been widely studied in pointfree topology and domain theory (see, e.g., [GHK+03, PP12]). A well-developed duality theory exists for the category AlgFrm of algebraic frames and its various subcategories, including the categories of arithmetic frames (also known as M-frames), coherent frames, and Stone frames. Indeed, a frame L is algebraic iff it is the frame of opens of a compactly based sober space X (see, e.g., [GHK+03, p. 423]). Furthermore, L is arithmetic iff X is stably compactly based, L is coherent iff X is spectral, and L is a Stone frame iff X is a Stone space (see Section 11 for details).

The duality theory for algebraic frames can be seen as a restriction of the well-known Hofmann–Lawson duality. The relationship between categories of continuous frames and algebraic frames is illustrated in Fig. 8.

$$\begin{array}{ccccccc}
\mathsf{ConFrm} & \hookleftarrow & \mathsf{StCFrm} & \hookleftarrow & \mathsf{StKFrm} & \hookleftarrow & \mathsf{KRFrm} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathsf{AlgFrm} & \hookleftarrow & \mathsf{AriFrm} & \hookleftarrow & \mathsf{CohFrm} & \hookleftarrow & \mathsf{StoneFrm}
\end{array}$$

Figure 8: Inclusion relationships between categories of continuous and algebraic frames.

In the previous chapter, we established Priestley duality for ConFrm and its subcategories listed in the first row of Fig. 8. The resulting dual equivalences are summarized in Fig. 7 at the end of the thesis. In this chapter, we further develop Priestley duality for AlgFrm and its subcategories listed in the second row of Fig. 8. To achieve this, we characterize the Priestley spaces of algebraic, coherent, arithmetic, and Stone frames.

To describe these frames in the language of Priestley spaces, we continue the approach of defining appropriate kernels (see Definition 8.1(1)). We introduce the *algebraic kernel* (`alg`), which characterizes algebraic frames, and the *zero-dimensional kernel* (`zer`), which describes zero-dimensional frames. These kernels can be compared to `con` and `reg` from the previous chapter, and we will analyze their relationships in detail. This highlights structural parallels among these different classes of L-spaces, and hence their corresponding frames.

The main results of this chapter establish new categories of L-spaces that are dually equivalent to the categories algebraic frames and are equivalent to important categories of topological spaces, such as spectral and Stone spaces.

The chapter is organized as follows. In Section 11, we describe the above categories of algebraic frames and their dual compactly based sober spaces. In Section 12, we introduce the kernel alg and characterize Priestley spaces of algebraic frames. This leads to a new proof of the duality between AlgFrm and KBSob. In Section 13, we characterize the Priestley spaces of arithmetic and coherent frames, yielding a new proof of their duality with compactly based spaces. Finally, in Section 14, we use the kernel zer to describe zero-dimensionality and characterize the Priestley spaces of Stone frames. We conclude by relating the Priestley spaces of coherent and Stone frames to Priestley duality for bounded distributive lattices and Stone duality for Boolean algebras.

11 Compactly based sober spaces and algebraic frames

In this section, we provide the necessary definitions and introduce the categories of algebraic frames and compactly based spaces mentioned in the introduction to this chapter. We then state the known dualities for these categories, as introduced earlier.

Let L be a frame. We write $\mathrm{K}(L)$ for the collection of compact elements of L (elements a satisfying $a \ll a$) and $\mathrm{C}(L)$ for the collection of complemented elements of L (elements a satisfying $a \vee a^* = 1$).

Definition 11.1.

(1) ([PP12, p. 142]) A frame L is *algebraic* if every element is a join of compact elements, i.e., $a = \bigvee\{b \in \mathrm{K}(L) \mid b \leq a\}$ for all $a \in L$.

(2) ([Joh82, p. 64]) A frame homomorphism $h\colon L \to M$ is *coherent* if it preserves compact elements, i.e., $a \in \mathrm{K}(L)$ implies $h(a) \in \mathrm{K}(M)$.

(3) Let AlgFrm be the category of algebraic frames and coherent frame homomorphisms.

Remark 11.2. Every algebraic frame is continuous, and a frame homomorphism between coherent frames is coherent iff it is proper (see, e.g., [Ban81, p. 4]). Consequently, AlgFrm is a full subcategory of ConFrm.

Definition 11.3.

(1) ([GHK+03, p. 117]) A frame L is *arithmetic* if it is algebraic and the way-below relation $\ll$ is stable.

(2) Let AriFrm be the full subcategory of AlgFrm consisting of arithmetic frames.

Remark 11.4.

(1) In [GHK+03], a lattice is defined as arithmetic if the binary meet of compact elements is compact. For algebraic lattices this is equivalent to $\ll$ being stable (see, e.g. [GHK+03, Prop I-4.8]).

(2) Arithmetic frames are also known as M-frames (see, e.g., [IM09]).

Definition 11.5.

(1) ([Joh82, p. 63–64]) A frame is *coherent* if it is both arithmetic and compact.

(2) Let CohFrm be the full subcategory of AriFrm consisting of coherent frames.

The next definition is well known (see, e.g., [Joh82, Ban89, Jak13]). We thank Joanne Walters-Wayland for informing us that the terminology of Stone frames originated from Banaschewski's University of Cape Town lecture notes (1988).

Definition 11.6.

(1) A frame L is *zero-dimensional* if every element is a join of complemented elements, i.e., $a = \bigvee\{b \in \mathrm{C}(L) \mid b \le a\}$ for all $a \in L$.

(2) A *Stone frame* is a zero-dimensional frame that is also compact.

(3) Let StoneFrm be the full subcategory of Frm consisting of Stone frames.

Remark 11.7. StoneFrm is clearly a full subcategory of KRFrm. Moreover, since every frame homomorphism preserves the well-inside relation $\prec$, and $\prec$ coincides with $\ll$ in compact regular frames, it follows that StoneFrm is a full subcategory of CohFrm.

Fig. 8 illustrates the relationship between categories of algebraic and continuous frames. See Table 2 for an overview of all the categories of frames defined in this thesis. We now consider the corresponding categories of topological spaces.

Recall that a topological space is compactly based if it has a basis of compact opens (see Item (1)) and that a continuous map is coherent if it pulls compact opens back to compact opens (see Definition 2.14).

Definition 11.8.

(1) Let KBSob denote the category of compactly based sober spaces with coherent continuous maps.

(2) A compactly based space X is *stably compactly based* if it is sober and the intersection of any two compact open sets is compact.

(3) Let StKBSp be the full subcategory of KBSob consisting of stably compactly based spaces.

Recall the definitions of spectral spaces (Definition 2.12) and Stone spaces (Definition 2.6) from Chapter I.

Remark 11.9.

(1) Spectral spaces are stably compactly based spaces that are also compact. Therefore, Spec is a full subcategory of StKBSp.

(2) KBSob is a full subcategory of LKSob since a continuous map between compactly based sober spaces is coherent iff it is proper. One implication is clear. The other follows from the fact that in a compactly based space X, every compact saturated set is an intersection of compact open sets. To see this, let $K \subseteq X$ be compact saturated. It suffices to show that for every $x \notin K$, there exists a compact open set U such that $K \subseteq U$ and $x \notin U$. Since X is compactly based for each $y \in K$ there is a compact open set U_y such that $y \in U_y$ and $x \notin U_y$. Thus, $K \subseteq \bigcup\{U_y \mid y \in K\}$. By the compactness of K and the fact that finite unions of compact sets remain compact, there exists a compact open set U such that $K \subseteq U$ and $x \notin U$.

The diagram in Fig. 9 illustrates the correspondence between categories of locally compact and compactly based sober spaces. A complete overview of the categories of topological spaces defined in this thesis can be found in Table 3.

$$\begin{array}{ccccccc}
\mathsf{LKSob} & \hookleftarrow & \mathsf{StLKSp} & \hookleftarrow & \mathsf{StKSp} & \hookleftarrow & \mathsf{KHaus} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathsf{KBSob} & \hookleftarrow & \mathsf{StKBSp} & \hookleftarrow & \mathsf{Spec} & \hookleftarrow & \mathsf{Stone}
\end{array}$$

Figure 9: Inclusion relationships between categories of locally compact and compactly based sober spaces.

The dualities for continuous frames and their subcategories, including those of Hofmann–Lawson and Isbell (see Theorem 7.7(1)), restrict to dualities for algebraic frames and compactly based spaces, along with their respective subcategories.

Theorem 11.10.

(1) AlgFrm *is dually equivalent to* KBSob.

(2) AriFrm *is dually equivalent to* StKBSp.

(3) CohFrm *is dually equivalent to* Spec.

(4) StoneFrm *is dually equivalent to* Stone.

One of the earliest references for Theorem 11.10 is [HK72, Thm. 5.7] (see also [GHK+03, p. 423]), which states the dualities for AlgFrm, AriFrm, and CohFrm. The duality for CohFrm is also explored in [Ban80, Ban81] and [Joh82, Sec. II.3]. This restricts further to the duality for StoneFrm (see, e.g., [Ban89] or [Jak13, Ch. IV]).

We thus arrive at the diagram in Fig. 10.

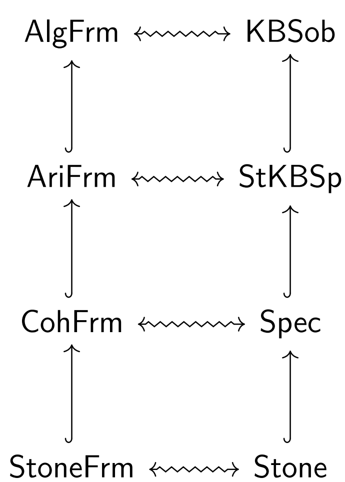

Figure 10: Correspondence between categories of algebraic frames and compactly based spaces.

Remark 11.11. It is straightforward to show that a continuous frame is algebraic iff its space of points is compactly based. This observation, together with Theorem 7.7 and the fact that AlgFrm and KBSob are full subcategories of ConFrm and LKSob, respectively, yields Theorem 11.10. However, a direct proof of Theorem 11.10 can be given without relying on Theorem 7.7. This follows from the fact that every algebraic frame is spatial. Since coherent frame homomorphisms correspond to coherent continuous maps, we can then restrict the equivalence of Theorem 1.10 to algebraic frames and compactly based spaces.

We now sketch the argument that every algebraic frame is spatial. Let L be an algebraic frame. Then Scott-open filters separate the elements of L. To see this, suppose $a \not\leq b$. Then there exists $k \in \mathrm{K}(L)$ such that $k \leq a$ but $k \not\leq b$. Therefore, ${\uparrow}k$ is a Scott-open filter such that $a \in {\uparrow}k$ and $b \notin k$. Finally, by the Prime Ideal Theorem, L is spatial iff Scott-open filters separate its elements (see Corollary 6.9(2)).

12 Priestley spaces of algebraic frames

In this section, we characterize algebraic frames using Priestley spaces. We then relate the Priestley duals of algebraic frames to compactly based sober spaces, leading to the above duality between AlgFrm and KBSob (see Theorem 11.10(1)).

Definition 12.1. Let X be an L-space.

(1) $\mathrm{ClopSUp}(X)$ denotes the collection of all clopen Scott upsets of X.

(2) For $U \in \mathrm{ClopUp}(X)$, define the *algebraic part* of U by

$$\mathtt{alg}\, U = \bigcup\{V \in \mathrm{ClopSUp}(X) \mid V \subseteq U\}.$$

If $U = \varphi(a)$, we simply write $\mathtt{alg}(a)$ for $\mathtt{alg}\, U$.

(3) X is L-*algebraic* if $\mathtt{alg}$ is representative.

Remark 12.2.

(1) The algebraic part of a clopen upset U is referred to as the core of U in [BM25, BBM25].

(2) It follows immediately from the definition that $\mathtt{alg}$ is a kernel, hence the kernel-related terminology in Definition 12.1(3) is justified.

We now establish the connection between algebraic frames and algebraic L-spaces.

Theorem 12.3. *A frame L is algebraic iff its Priestley space X is* L*-algebraic.*

Proof. Let $a \in L$. By (I.3),

$$\varphi\left(\bigvee S\right) = \operatorname{cl} \bigcup \varphi[S]$$

for each $S \subseteq L$. Therefore, by Theorem 5.8(1), we have that $a = \bigvee \{b \in \mathrm{K}(L) \mid b \leq a\}$ iff

$$\varphi(a) = \operatorname{cl} \bigcup \{\varphi(b) \in \mathrm{ClopSUp}(X) \mid \varphi(b) \subseteq \varphi(a)\} = \operatorname{cl} \mathtt{alg}(a).$$

Thus, L is algebraic iff $\mathtt{alg}$ is representative. □

In Proposition 12.5, we establish several conditions equivalent to being L-algebraic in the setting of continuous L-spaces. For this we require the following lemma.

Lemma 12.4. *Let X be an* L*-space.*

(1) $\mathtt{alg} \leq \mathtt{con}$.

(2) *If X is* L*-algebraic, then $\mathtt{alg} = \mathtt{con}$ and hence X is* L*-continuous.*

Proof. (1) Suppose $U \in \mathrm{ClopUp}(X)$ and $x \in \mathtt{alg}\, U$. Then there exists $V \in \mathrm{ClopSUp}(X)$ such that $x \in V \subseteq U$. Let $W \in \mathrm{OpUp}(X)$ be such that $U \subseteq \operatorname{cl} W$. Then $V \subseteq \operatorname{cl} W$, so $V \subseteq W$ by Lemma 5.2(3). Hence, $V \ll U$. Therefore, $x \in \mathtt{con}\, U$, which implies that $\mathtt{alg}\, U \subseteq \mathtt{con}\, U$.

(2) By (1), it suffices to show that $\texttt{con} \leq \texttt{alg}$. Let $U \in \text{ClopUp}(X)$ and $x \in \texttt{con}\, U$. Then there exists $V \in \text{ClopUp}(X)$ such that $x \in V \ll U$. Since $\texttt{alg}$ is representative, $U = \text{cl}\, \texttt{alg}\, U$, and hence $x \in V \subseteq \texttt{alg}\, U$ because $V \ll U$. Therefore, $\texttt{alg} = \texttt{con}$, and hence $\texttt{con}$ is representative, making X L-continuous. □

Proposition 12.5. *For a continuous* L*-space X, the following conditions are equivalent:*

(1) $\texttt{con} = \texttt{alg}$.

(2) $\texttt{alg}$ *is representative.*

(3) *For each $U \in \text{ClopUp}(X)$ and every $y \in U \cap \text{loc}\, X$, there exists $V \in \text{ClopSUp}(X)$ such that $y \in V \subseteq U$.*

(4) *For each $U \in \text{ClopUp}(X)$ and every $F \in \text{SUp}(X)$ with $F \subseteq \texttt{con}\, U$, there exists $V \in \text{ClopSUp}(X)$ such that $F \subseteq V \subseteq U$.*

Proof. (1)⇒(2) Since X is L-continuous, $\texttt{con}$ is representative. Therefore, (1) implies that $\texttt{alg}$ is representative.

(2)⇒(3) Let $U \in \text{ClopUp}(X)$ and suppose that $y \in U \cap \text{loc}\, X$. By (2), $U = \text{cl}\, \texttt{alg}\, U$, which implies that $y \in \text{cl}(\texttt{alg}\, U) \cap \text{loc}\, X$. By Lemma 4.16(1), $\text{cl}(\texttt{alg}\, U) \cap \text{loc}\, X = \texttt{alg}\, U \cap \text{loc}\, X$. Therefore, $y \in \texttt{alg}\, U$, and so there is $V \in \text{ClopSUp}(X)$ such that $y \in V \subseteq U$.

(3)⇒(4) Let $U \in \text{ClopUp}(X)$ and $F \subseteq \texttt{con}\, U$ be a Scott upset. Suppose $y \in F \cap \text{loc}\, X$. Then $y \in \texttt{con}\, U$, so $y \in U$. Therefore, by (3), there is $V_y \in \text{ClopSUp}(X)$ such that $y \in V_y \subseteq U$. Thus,

$$F = \bigcup\{{\uparrow}y \mid y \in F \cap \text{loc}\, X\} \subseteq \bigcup\{V_y \mid y \in F \cap \text{loc}\, X\} \subseteq U.$$

Because F is closed, it is compact. Hence, since a finite union of clopen Scott upsets is a clopen Scott upset, there exists $V \in \text{ClopSUp}(X)$ such that $F \subseteq V \subseteq U$.

(4)⇒(1) By Lemma 12.4(1), $\mathtt{alg} \le \mathtt{con}$. For the reverse inclusion, it suffices to show that for all $U, V \in \mathrm{ClopUp}(X)$, $V \ll U$ implies that there is $W \in \mathrm{ClopSUp}(X)$ with $V \subseteq W \subseteq U$. Let $V \ll U$. By Proposition 8.13(2), there is a Scott upset F such that $V \subseteq F \subseteq U$. Since $U = \mathrm{cl}(\mathtt{con}\, U)$, Lemma 5.2(3) yields that $F \subseteq \mathtt{con}\, U$. Therefore, by (4), there exists $W \in \mathrm{ClopSUp}(X)$ such that $F \subseteq W \subseteq U$, and hence $V \subseteq W \subseteq U$. □

To extend the correspondence between algebraic L-spaces and algebraic frames to a categorical equivalence, we now examine morphisms between algebraic L-spaces.

Definition 12.6.

(1) An L-morphism $f : X \to Y$ between L-spaces is *coherent* if

$$f^{-1}(\mathtt{alg}\, U) \subseteq \mathtt{alg}\, f^{-1}(U) \qquad \text{for all } U \in \mathrm{ClopUp}(Y).$$

(2) Let $\mathsf{AlgLPries}$ be the category of algebraic L-spaces and coherent L-morphisms.

It is easy to see that the identity morphism is a coherent L-morphism and that the composition of two coherent L-morphisms is coherent. Therefore, $\mathsf{AlgLPries}$ is well defined as a category. To establish that $\mathsf{AlgLPries}$ is a full subcategory of $\mathsf{ConLPries}$, we first prove the following:

Lemma 12.7. *Let $f\colon X \to Y$ be an* L*-morphism between* L*-spaces.*

(1) *If f is proper and X is an algebraic* L*-space, then f is coherent.*

(2) *If f is coherent and Y is an algebraic* L*-space, then f is proper.*

(3) *If X and Y are algebraic* L*-spaces, then f is coherent iff f is proper.*

Proof. (1) Let $U \in \mathrm{ClopUp}(Y)$. Then

$$
\begin{aligned}
f^{-1}(\mathtt{alg}\, U) &\subseteq f^{-1}(\mathtt{con}\, U) && \text{by Lemma 12.4(1)} \\
&\subseteq \mathtt{con}\, f^{-1}(U) && \text{since } f \text{ is proper} \\
&= \mathtt{alg}\, f^{-1}(U) && \text{by Lemma 12.4(2) and Proposition 12.5(1).}
\end{aligned}
$$

(2) Let $U \in \mathrm{ClopUp}(Y)$. Then

$$
\begin{aligned}
f^{-1}(\mathtt{con}\, U) &= f^{-1}(\mathtt{alg}\, U) && \text{by Lemma 12.4(2) and Proposition 12.5(1)} \\
&\subseteq \mathtt{alg}\, f^{-1}(U) && \text{since } f \text{ is coherent} \\
&\subseteq \mathtt{con}\, f^{-1}(U) && \text{by Lemma 12.4(1).}
\end{aligned}
$$

(3) This follows from (1) and (2). □

We thus arrive at the following result:

Proposition 12.8. AlgLPries *is a full subcategory of* ConLPries.

Proof. Apply Lemmas 12.4(2) and 12.7(3). □

We now establish the first main result of this section: the dual equivalence between the categories of algebraic frames and algebraic L-spaces.

Theorem 12.9. AlgFrm *is dually equivalent to* AlgLPries.

Proof. By Remark 11.2, AlgFrm is a full subcategory of ConFrm. By Proposition 12.8, AlgLPries is a full subcategory of ConLPries. Thus, the result follows from Theorems 8.9 and 12.3. □

Next, we establish the connection between $\mathsf{AlgLPries}$ and the category KBSob of compactly based sober spaces. The following is a characterization of compact open sets of the localic part of a spatial L-space. This result relies on Theorem 6.7, which established an analogue of the Hofmann–Mislove Theorem for Priestley spaces by relating Scott upsets of an L-space to compact saturated sets of its localic part.

Proposition 12.10. *Let X be a spatial* L*-space and $U \subseteq X$. Then $U \in \mathrm{ClopSUp}(X)$ iff there is a compact open set V of $\mathrm{loc}\, X$ such that $\mathrm{cl}\, V = U$.*

Proof. First, suppose that $U \in \mathrm{ClopSUp}(X)$. Then $V \coloneqq U \cap \mathrm{loc}\, X$ is a compact saturated subset of $\mathrm{loc}\, X$ by Theorem 6.7. Moreover, V is an open subset of $\mathrm{loc}\, X$ since $U \in \mathrm{ClopUp}(X)$. Furthermore, $\mathrm{cl}\, V = U$ because X is a spatial L-space (see Theorem 4.4).

Conversely, suppose there is a compact open set V of $\mathrm{loc}\, X$ such that $\mathrm{cl}\, V = U$. Then ${\uparrow}V$ is a Scott upset of X by Theorem 6.7. Since V is open and X is a spatial L-space, there is $U' \in \mathrm{ClopUp}(X)$ such that $V = U' \cap \mathrm{loc}\, X$ and $\mathrm{cl}\, V = U'$ (see Theorem 4.4). Therefore, $U = \mathrm{cl}\, V = U'$, and so $U \in \mathrm{ClopUp}(X)$. Moreover,

$$U = {\uparrow}U = {\uparrow}\,\mathrm{cl}\, V = \mathrm{cl}\,{\uparrow}V = {\uparrow}V,$$

where the third equality follows from Lemma 3.4(4). Thus, U is a Scott upset. □

Theorem 12.11. *Let X be an* L*-space. If X is* L*-algebraic, then $\mathrm{loc}\, X$ is a compactly based sober space. If X is* L*-spatial, then the converse holds.*

Proof. First, suppose that X is an algebraic L-space. Then $\mathrm{loc}\, X$ is sober by Proposition 4.14. Let $V \subseteq \mathrm{loc}\, X$ be open and $y \in V$. Set $U \coloneqq \mathrm{cl}\, V$. Then $U \in \mathrm{ClopUp}(X)$ by Remark 4.10(2).

Moreover, by Lemma 4.16(2),

$$U \cap \operatorname{loc} X = \operatorname{cl} V \cap \operatorname{loc} X = V,$$

so $y \in U \cap \operatorname{loc} X$. By Lemma 12.4(2) and Proposition 12.5(3), there is $W \in \operatorname{ClopSUp}(X)$ such that $y \in W \subseteq U$. Therefore,

$$y \in W \cap \operatorname{loc} X \subseteq U \cap \operatorname{loc} X = V,$$

where $W \cap \operatorname{loc} X$ is a compact open subset of $\operatorname{loc} X$ by Proposition 12.10 (L-algebraic implies L-spatial by Proposition 8.14 and Lemma 12.4(2)). Thus, $\operatorname{loc} X$ is compactly based.

Conversely, suppose that X is L-spatial and $\operatorname{loc} X$ is compactly based. We will show that `alg` is representative. Let $U \in \operatorname{ClopUp}(X)$. Since compactly based spaces are locally compact, X is L-continuous by Theorem 8.15. Therefore, by Proposition 12.5(3), it suffices to show that for each $y \in U \cap \operatorname{loc} X$ there is $V \in \operatorname{ClopSUp}(X)$ such that $y \in V \subseteq U$. Because $U \cap \operatorname{loc} X$ is an open subset of $\operatorname{loc} X$ and $\operatorname{loc} X$ is compactly based, there is a compact open $K \subseteq \operatorname{loc} X$ such that $y \in K \subseteq U \cap \operatorname{loc} X$. Therefore, $\operatorname{cl} K \in \operatorname{ClopSUp}(X)$ by Proposition 12.10. Moreover, $y \in \operatorname{cl} K \subseteq \operatorname{cl}(U \cap \operatorname{loc} X) = U$. Thus, X is L-algebraic. ☐

By Proposition 12.8, AlgLPries is a full subcategory of ConLPries. By Remark 11.9(2), KBSob is a full subcategory of LKSob. Thus, as an immediate consequence of Theorems 8.20 and 12.11, we obtain the desired equivalence:

Corollary 12.12. AlgLPries *is equivalent to* KBSob*.*

13 Priestley spaces of arithmetic and coherent frames: Priestley revisited

In this section, we describe Priestley duals of arithmetic and coherent frames. These L-spaces are distinguished by the stability of the kernel `alg`. We further establish their connections to stably compactly based and spectral spaces, leading to alternative proofs of Theorem 11.10(2,3).

Definition 13.1.

(1) An algebraic L-space is L-*arithmetic* if `alg` is stable.

(2) Let AriLPries be the full subcategory of AlgLPries consisting of arithmetic L-spaces.

Remark 13.2. In [BM25], arithmetic L-spaces are alternatively defined as algebraic L-spaces in which `con` is stable. This definition is equivalent to the one given above by Proposition 13.3.

The following proposition gives alternative characterizations of arithmetic L-spaces.

Proposition 13.3. *For an algebraic* L*-space* X*, the following conditions are equivalent:*

(1) X *is* L*-arithmetic.*

(2) `con` *is stable.*

(3) $\mathrm{SUp}(X)$ *is closed under binary intersections.*

(4) $\mathrm{ClopSUp}(X)$ *is closed under binary intersections.*

Proof. (1)⇒(2) This follows directly from Proposition 12.5(1).

(2)⇒(3) This is given by Lemma 9.6(2).

(3)⇒(4) This is immediate since $\mathrm{ClopSUp}(X)$ is a subset of $\mathrm{SUp}(X)$ and binary intersections of clopens are clopen.

(4)⇒(1) Suppose $U_1, U_2 \in \mathrm{ClopUp}(X)$. By Lemma 2.8(2), it suffices to show that for each $W \in \mathrm{ClopUp}(X)$, we have $W \subseteq \mathtt{alg}\, U_1 \cap \mathtt{alg}\, U_2$ iff $W \subseteq \mathtt{alg}(U_1 \cap U_2)$. Note that since W is compact, $W \subseteq \mathtt{alg}\, U_1$ iff there exists $V_1 \in \mathrm{ClopSUp}(X)$ such that $W \subseteq V \subseteq \mathtt{alg}\, U_1$.

Therefore, it follows from (4) that

$$
\begin{aligned}
W \subseteq \mathtt{alg}\, U_1 \cap \mathtt{alg}\, U_2 &\iff \exists V_1, V_2 \in \mathrm{ClopSUp}(X) : W \subseteq V_1 \subseteq U_1 \text{ and } W \subseteq V_2 \subseteq U_2 \\
&\iff \exists V \in \mathrm{ClopSUp}(X) : W \subseteq V \subseteq U_1 \cap U_2 \\
&\iff W \subseteq \mathtt{alg}(U_1 \cap U_2). \qquad \square
\end{aligned}
$$

To connect arithmetic frames, arithmetic L-spaces, and stably compactly based spaces, we need the following fact.

Lemma 13.4. *Let X be a compactly based sober space. Then X is stably locally compact iff X is stably compactly based.*

Proof. The left-to-right implication is immediate. For the other implication, let $A, B \subseteq X$ be compact saturated. Since X is compactly based, each compact saturated set can be written as an intersection of compact open sets (see Remark 11.9(2)). Thus, $A \cap B = \bigcap \mathcal{F}$, where

$$\mathcal{F} = \{U \cap V \mid U, V \text{ compact open with } A \subseteq U \text{ and } B \subseteq V\}.$$

Because X is stably compactly based, $\mathcal{F}$ is down directed. Therefore, the Hofmann–Mislove Theorem implies that $\bigcap \mathcal{F}$ is compact (see, e.g., [GHK+03, Cor. II-1.22]). Consequently, $A \cap B$ is compact. $\square$

Theorem 13.5. *For an algebraic frame L and its Priestley space X, the following conditions are equivalent:*

1. *L is an arithmetic frame.*
2. *X is an arithmetic* L-*space.*
3. loc X *is a stably compactly based space.*

Proof. By Theorem 12.3, L being algebraic implies that X is L-algebraic. Therefore, by Theorem 12.11, $\operatorname{loc} X$ is a compactly based sober space.

(1)⇔(2) Assume L is an arithmetic frame and $\varphi(a), \varphi(b) \in \mathrm{ClopSUp}(X)$. The latter implies $a, b \in \mathrm{K}(L)$ by Theorem 5.8(1). Since L is an arithmetic frame, $a \wedge b \in \mathrm{K}(L)$. It follows that $\varphi(a) \cap \varphi(b) = \varphi(a \wedge b)$ is a Scott upset, again by Theorem 5.8(1). Thus, X is an arithmetic L-space by Proposition 13.3.

Conversely, let X be an arithmetic L-space and $a, b \in \mathrm{K}(L)$. By Theorem 5.8(1), $\varphi(a), \varphi(b)$ are clopen Scott upsets. Therefore, by Proposition 13.3, $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$ is a Scott upset. Thus, $a \wedge b \in \mathrm{K}(L)$, again by Theorem 5.8(1). Hence, L is an arithmetic frame.

(2)⇔(3) Since X is L-algebraic (and hence L-continuous), X is L-arithmetic iff it is L-stably continuous by Proposition 13.3. But X is a stably continuous L-space iff $\operatorname{loc} X$ is a stably locally compact space by Theorem 9.7(1). Since $\operatorname{loc} X$ is a compactly based sober space, $\operatorname{loc} X$ is stably locally compact iff it is stably compactly based by Lemma 13.4. Thus, X is an arithmetic L-space iff $\operatorname{loc} X$ is a stably compactly based space. □

The following theorem establishes categorical equivalences between arithmetic L-spaces, arithmetic frames, and stably compactly based spaces.

Theorem 13.6.

(1) $\mathsf{AriLPries}$ *is dually equivalent to* AriFrm.

(2) $\mathsf{AriLPries}$ *is equivalent to* StKBSp.

Proof. (1) follows from Theorems 12.9 and 13.5, while (2) from Corollary 12.12 and Theorem 13.5 □

This provides an alternative proof of the well-known duality for arithmetic frames (see Theorem 11.10(2)):

Corollary 13.7. AriFrm *is dually equivalent to* StKBSp.

We now focus on the Priestley duals of coherent frames. Since coherent frames are compact arithmetic frames, it follows that their Priestley duals are compact arithmetic L-spaces (see Theorem 5.8(2)). We also establish a connection between compact arithmetic L-spaces and spectral spaces, leading to the well-known duality between CohFrm and Spec (see Theorem 11.10(3)).

Definition 13.8.

(1) A compact arithmetic L-space is called L-*coherent*.

(2) Let CohLPries denote the full subcategory of AriLPries consisting of coherent L-spaces.

Theorem 13.9. *For an algebraic frame L and its Priestley space X, the following conditions are equivalent:*

(1) *L is a coherent frame.*

(2) *X is a coherent* L*-space.*

(3) $\operatorname{loc} X$ *is a spectral space.*

Proof. (1)⇔(2) L is coherent iff it is a compact arithmetic frame. By Theorems 5.8(2) and 13.5, this holds iff X is L-coherent.

(2)⇔(3) By Lemma 5.12 and Theorem 13.5, X is L-coherent iff $\operatorname{loc} X$ is a compact stably compactly based space, which is precisely a spectral space. □

The following corollary, obtained from Theorems 13.6 and 13.9, establishes the last result of this section.

Corollary 13.10.

(1) $\mathsf{CohLPries}$ *is dually equivalent to* CohFrm.

(2) $\mathsf{CohLPries}$ *is equivalent to* Spec.

To conclude this section, we highlight a connection to Priestley duality for bounded distributive lattices. The equivalence of the categories DLat of bounded distributive lattices and CohFrm of coherent frames (see Remark 6.4(1)) implies, via Theorem 13.10(1), that DLat is dually equivalent to $\mathsf{CohLPries}$. Moreover, by Theorem 13.10(2), $\mathsf{CohLPries}$ is equivalent to Spec. By Theorem 2.17, Spec is isomorphic to Pries. This provides a new perspective on Priestley duality through the framework of L-spaces (see the first row of Fig. 12 at the end of the next section).

14 Priestley spaces of Stone frames: the unification of kernels

In the final section of this chapter, we characterize the Priestley duals of Stone frames. A compact frame is Stone if it has sufficiently many complemented elements. From the perspective of Priestley spaces, complemented elements correspond to clopen bisets (see, e.g., [BGJ16, Lem 6.1]). Recall (see Section 10) that a biset is a subset that is both an upset and a downset.

Definition 14.1. Let X be an L-space.

(1) Let $\mathrm{ClopBi}(X)$ denote the collection of clopen bisets of X.

(2) For $U \in \mathrm{ClopUp}(X)$, define the *zero-dimensional part of* U by

$$\mathtt{zer}\, U = \bigcup \{V \in \mathrm{ClopBi}(X) \mid V \subseteq U\}.$$

(3) An L-space is L-*zero-dimensional* if $\mathtt{zer}$ is representative.

(4) An L-space is L-*Stone* if it is both L-compact and L-zero-dimensional.

(5) Let StoneLPries be the full subcategory of LPries consisting of Stone L-spaces.

Remark 14.2.

(1) The zero-dimensional part of a clopen upset U is referred to as the biregular part in [BGJ16], and as the center in [BM25].

(2) It follows immediately from the definition that $\mathtt{zer}$ is a kernel, hence the kernel-related terminology in Definition 14.1(3) is justified.

Lemma 14.3. *Let X be an* L*-space.*

(1) $\mathtt{zer}$ *is a kernel.*

(2) $\mathtt{zer} \leq \mathtt{reg}$.

(3) *If X is* L*-zero-dimensional, then X is* L*-regular.*

(4) *If X is a Stone* L*-space, then X is a compact regular* L*-space.*

Proof. (1) This follows directly from the definition of the zero-dimensional part.

(2) Suppose $x \in \mathtt{zer}\, U$. Then there exists $V \in \mathrm{ClopBi}(X)$ such that $x \in V \subseteq U$. Therefore, ${\downarrow}{\uparrow}x \subseteq U$, which implies $x \in \mathtt{reg}\, U$ by Lemma 10.4(1).

(3) Suppose X is L-zero-dimensional. Since $\mathtt{zer}$ is representative and $\mathtt{zer} \leq \mathtt{reg}$ by (2), it follows that $\mathtt{reg}$ is representative by (2), making X L-regular.

(4) Since X is L-compact, it follows from (3) that X is a compact regular L-space. □

This immediately yields the following result:

Proposition 14.4. StoneLPries *is a full subcategory of* KRLPries.

Next, we show that StoneLPries is also a full subcategory of CohLPries. For this we establish that the four kernels we considered all coincide in the setting of Stone L-spaces. We first require the L-space analogue of the result that each compact saturated set in a compactly based space is the intersection of the compact open sets containing it (see Remark 11.9(2)):

Lemma 14.5. *Let X be an algebraic* L*-space and $F \subseteq X$ a Scott upset. Then*

$$F = \bigcap \{V \in \mathrm{ClopSUp}(X) \mid F \subseteq V\}.$$

Proof. Since F is a closed upset, $F = \bigcap\{U \in \mathrm{ClopUp}(X) \mid F \subseteq U\}$ by Lemma 2.8(3). Thus, it is enough to show that for each $U \in \mathrm{ClopUp}(X)$ with $F \subseteq U$, there exists $V \in \mathrm{ClopSUp}(X)$ such that $F \subseteq V \subseteq U$. Since $\mathtt{alg}$ is representative, $U = \mathrm{cl}\,\mathtt{alg}\,U$, so $F \subseteq U$ implies that $F \subseteq \mathtt{alg}\,U$ by Lemma 5.2(3). By compactness, we obtain the desired V. □

Theorem 14.6. *Let X be a Stone* L*-space.*

(1) $\mathrm{ClopSUp}(X) = \mathrm{ClopBi}(X)$.

(2) $\mathtt{zer} = \mathtt{reg} = \mathtt{alg} = \mathtt{con}$.

Proof. (1) Since X is a Stone L-space, it follows from Lemma 14.3(4) that it is a compact regular L-space. Therefore, by Lemma 10.15(4), Scott upsets coincide with closed bisets, proving the claim.

(2) The inclusion $\mathtt{zer} \leq \mathtt{reg}$ follows from Lemma 14.3(2), while $\mathtt{alg} \leq \mathtt{con}$ follows from Lemma 12.4(1). Thus, it remains to show that $\mathtt{reg} \leq \mathtt{alg}$ and $\mathtt{con} \leq \mathtt{zer}$. We begin by proving that $\mathtt{reg} \leq \mathtt{alg}$. Suppose $U \in \mathrm{ClopUp}(X)$ and let $x \in \mathtt{reg}\,U$. Then there is $V \in \mathrm{ClopUp}(X)$ such that $x \in V$ and ${\downarrow}V \subseteq U$. Hence, ${\uparrow}{\downarrow}x \subseteq U$. Since X is L-compact, $\min({\downarrow}x) \subseteq \min X \subseteq \mathrm{loc}\,X$, and so ${\uparrow}{\downarrow}x$ is a Scott upset because it is closed. Since X is L-algebraic, $\mathtt{alg}$ is representative, so $\mathrm{cl}\,\mathtt{alg}\,U = U$. Thus, ${\uparrow}{\downarrow}x \subseteq \mathtt{alg}\,U$ by Lemma 5.2(3), which implies that $x \in \mathtt{alg}\,U$. Consequently, $\mathtt{reg} \leq \mathtt{alg}$.

To complete the proof, we show that $\texttt{con} \leq \texttt{zer}$. It suffices to show that for all $U, V \in \mathrm{ClopUp}(X)$, if $V \subseteq \texttt{con}\, U$ then $V \subseteq \texttt{zer}\, U$. Observe that

$$\begin{aligned}
V \subseteq \texttt{con}\, U &\implies V \ll U && \text{by Lemma 8.4(2)} \\
&\implies \exists F \in \mathrm{SUp}(X) : V \subseteq F \subseteq U && \text{by Proposition 8.13(2)} \\
&\implies \exists W \in \mathrm{ClopSUp}(X) : V \subseteq W \subseteq U && \text{by Lemma 14.5} \\
&\implies \exists W \in \mathrm{ClopBi}(X) : V \subseteq W \subseteq U && \text{by (1)} \\
&\implies V \subseteq \texttt{zer}\, U,
\end{aligned}$$

where the third implication follows from Proposition 13.3(4) since X is compact. □

Corollary 14.7. StoneLPries *is a full subcategory of* CohLPries.

Proof. By Theorem 14.6(2), every Stone L-space is a coherent L-space. Furthermore, since StoneLPries is a full subcategory of KRLPries, every L-morphism between Stone L-spaces is a proper L-morphism by Proposition 10.18(2). Therefore, every such morphism is also a coherent L-morphism by Lemma 12.7(3). Thus, StoneLPries is a full subcategory of CohLPries. □

[BGJ16, Thm. 6.3(1)] establishes that Priestley duals of zero-dimensional frames are precisely zero-dimensional L-spaces. We establish a connection between zero-dimensional L-spaces and zero-dimensional topological spaces.

Lemma 14.8. *Let X be an* L*-space.*

(1) *If $U \in \mathrm{ClopBi}(X)$, then $U \cap \mathrm{loc}\, X$ is a clopen subset of $\mathrm{loc}\, X$.*

(2) *If X is* L*-spatial and $V \subseteq \mathrm{loc}\, X$ is clopen, then there exists $U \in \mathrm{ClopBi}(X)$ such that $V = U \cap \mathrm{loc}\, X$.*

Proof. (1) This follows directly from the definition of the topology on $\operatorname{loc} X$ (see Definition 4.8).

(2) Let $V \subseteq \operatorname{loc} X$ be clopen. Since V is open, there exists $U \in \operatorname{ClopUp}(X)$ such that $V = U \cap \operatorname{loc} X$ and $\operatorname{cl} V = U$ (see Theorem 4.4). Similarly, since V is closed, there exists $W \in \operatorname{ClopUp}(X)$ such that $\operatorname{loc} X \setminus V = W \cap \operatorname{loc} X$ and $\operatorname{cl}(\operatorname{loc} X \setminus V) = W$. Since $V, \operatorname{loc} X \setminus V$ are open in $\operatorname{loc} X$, we have

$$U \cap W = \operatorname{cl} V \cap \operatorname{cl}(\operatorname{loc} X \setminus V) = \operatorname{cl}(V \cap (\operatorname{loc} X \setminus V)) = \varnothing$$

by Lemma 4.16(3). Moreover, by Theorem 4.4,

$$U \cup W = \operatorname{cl} V \cup \operatorname{cl}(\operatorname{loc} X \setminus V) = \operatorname{cl}(V \cup (\operatorname{loc} X \setminus V)) = \operatorname{cl} \operatorname{loc} X = X.$$

Thus, $U = X \setminus W$, implying that $U \in \operatorname{ClopBi}(X)$. □

Proposition 14.9. *Let X be an* L*-space. If X is* L*-zero-dimensional, then $\operatorname{loc} X$ is zero-dimensional. If in addition X is* L*-spatial, then the converse holds.*

Proof. First, suppose X is L-zero-dimensional. Let $V \subseteq \operatorname{loc} X$ be open and $y \in V$. Then there exists $U \in \operatorname{ClopUp}(X)$ such that $U \cap \operatorname{loc} X = V$. Since $\mathtt{zer}$ is representative, it follows that

$$U \cap \operatorname{loc} X = \operatorname{cl}(\mathtt{zer}\, U) \cap \operatorname{loc} X = \mathtt{zer}\, U \cap \operatorname{loc} X,$$

where the last equality follows from Lemma 4.16(1) because $\mathtt{zer}\, U$ is an open upset of X. Therefore, there exists $W \in \operatorname{ClopBi}(X)$ such that $y \in W \subseteq U$. Thus, $y \in W \cap \operatorname{loc} X \subseteq V$ and $W \cap \operatorname{loc} X$ is clopen in $\operatorname{loc} X$ by Lemma 14.8(1). Hence, $\operatorname{loc} X$ is zero-dimensional.

Conversely, suppose that X is L-spatial, $\operatorname{loc} X$ is zero-dimensional, and $U \in \operatorname{ClopUp}(X)$. Since X is L-spatial, $U \cap \operatorname{loc} X$ is dense in U (see Theorem 4.4). Therefore, it suffices to show that $U \cap \operatorname{loc} X \subseteq \mathtt{zer}\, U$. Let $y \in U \cap \operatorname{loc} X$. Since $U \in \operatorname{ClopUp}(X)$, we have that $U \cap \operatorname{loc} X$ is open in $\operatorname{loc} X$. Because $\operatorname{loc} X$ is zero-dimensional, there exists a clopen set $V \subseteq \operatorname{loc} X$ such that $y \in V \subseteq U \cap \operatorname{loc} X$. Since V is clopen in $\operatorname{loc} X$, Lemma 14.8(2) implies that there is $W \in \operatorname{ClopBi}(X)$ such that $V = W \cap \operatorname{loc} X$. Because X is L-spatial, we have $\operatorname{cl} V = W$, so $y \in W \subseteq U$. Thus, $y \in \mathtt{zer}\, U$. □

Theorem 14.10. *For a spatial frame L and its Priestley space X, the following conditions are equivalent:*

(1) *L is a zero-dimensional frame.*

(2) *X is a zero-dimensional* L*-space.*

If in addition L is spatial, then (1) *and* (2) *are equivalent to*

(3) $\operatorname{loc} X$ *is a zero-dimensional space.*

Proof. The equivalence (1)⇔(2) follows from [BGJ16, Thm. 6.3(1)]. For completeness, we provide a proof. Let $a \in L$. Since $b \in \mathrm{C}(L)$ iff $\varphi(b) \in \operatorname{ClopBi}(X)$,

$$a = \bigvee\{b \in \mathrm{C}(L) \mid b \leq a\} \iff \varphi(a) = \operatorname{cl}\bigcup\{\varphi(b) \in \operatorname{ClopBi}(X) \mid \varphi(b) \subseteq \varphi(a)\}$$
$$\iff \varphi(a) = \operatorname{cl} \mathtt{zer}(a)$$

by (I.3). Therefore, L is zero-dimensional iff $\mathtt{zer}$ is representative.

(2)⇔(3) This follows from Proposition 14.9. □

Corollary 14.11. *Let L be a frame and X its Priestley space.*

(1) *L is a Stone frame.*

(2) *X is a Stone* L*-space.*

If in addition L is spatial, then (1) *and* (2) *are equivalent to*

(3) $\operatorname{loc} X$ *is a Stone space.*

Proof. (1)⇔(2) This follows from Theorems 5.8(2) and 14.10.

(2)⇔(3) This follows from Lemma 5.12 and Theorem 14.10. □

As an immediate consequence, we arrive at the final results of this section.

Corollary 14.12.

(1) $\mathsf{StoneLPries}$ *is dually equivalent to* $\mathsf{StoneFrm}$.

(2) $\mathsf{StoneLPries}$ *is equivalent to* Stone.

Proof. These results follow from Corollaries 13.10 and 14.11 and the fact that $\mathsf{StoneFrm}$, $\mathsf{StoneLPries}$, and Stone are full subcategories of CohFrm, $\mathsf{CohLPries}$, and Spec, respectively (see Remark 11.7, Corollary 14.7, and Remark 2.15). □

This provides an alternative proof of the well-known duality for Stone frames (see Theorem 11.10(4)).

Corollary 14.13. $\mathsf{StoneFrm}$ *is dually equivalent to* Stone.

To summarize, the diagram in Fig. 11 presents the established equivalences, using the same notation as in previous diagrams. Table 1 provides an overview of all the introduced categories of Priestley spaces in this thesis. The corresponding categories of frames and spaces are summarized in Tables 2 and 3.

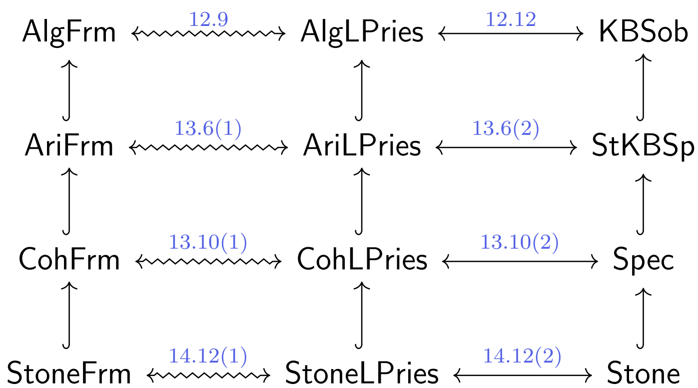

Figure 11: Equivalences and dual equivalences among categories of algebraic frames, algebraic L-spaces, and compactly based sober spaces.

At the end of Section 13, we described how our framework yields a new perspective on Priestley duality for bounded distributive lattices. Since the equivalence of DLat and CohFrm restricts to an equivalence between BA and StoneFrm (see, e.g., [Ban89, p. 258]), we also obtain a new view on Stone duality for Boolean algebras (see the second row of Fig. 12):

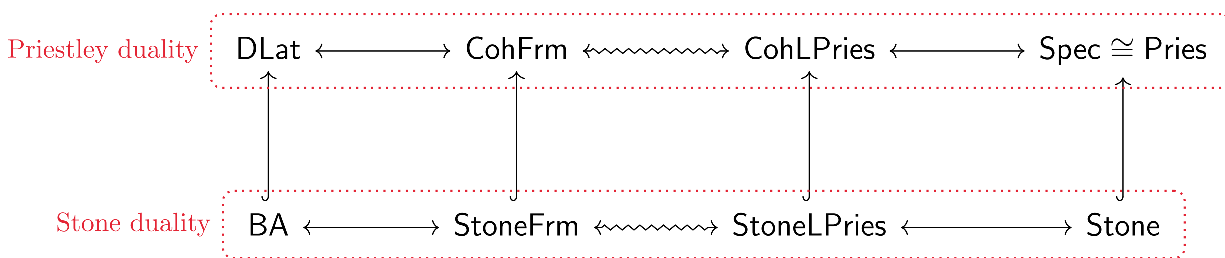

Figure 12: Priestley and Stone dualities revisited through the dual equivalence of the categories of coherent frames and coherent L-spaces.

We now examine how the topology of the corresponding Priestley space or Stone space emerges in the L-space. (For this it is useful to review the notation introduced in Remark 2.18.)

Remark 14.14.

(1) By the equivalence between DLat and CohFrm, every bounded distributive lattice D is

isomorphic to the lattice $\mathrm{K}(L)$ of compact elements of a coherent frame L. Let X be the Priestley space of L, and Y the Priestley space of D. Identifying D with $\mathrm{K}(L)$, the map $P \mapsto P \cap D$ establishes an isomorphism between $(\operatorname{loc} X, \subseteq)$ and $(Y, \subseteq)$. However, the topology π_Y of Y is different from the subspace topology $\pi_{\operatorname{loc} X}$ on $\operatorname{loc} X$ induced from π_X. Indeed, the topology of Y is the patch topology π_Y of the upper topology τ_Y. Identifying Y with $\operatorname{loc} X$, we obtain $\varphi_D(a) = \varphi_L(a) \cap \operatorname{loc} X$ for each $a \in D$. Since $\mathrm{ClopSUp}(X)$ corresponds to D (see Theorem 5.8(1)), π_Y is generated by the basis

$$\{(U \setminus V) \cap \operatorname{loc} X \mid U, V \in \mathrm{ClopSUp}(X)\}.$$

Thus, π_Y is the patch topology of the subspace topology $\tau_{\operatorname{loc} X}$ on $\operatorname{loc} X$ induced by the upper topology τ_X. Next, we show that this topology may differ from the subspace topology induced by π_X.

(2) If D is a Boolean algebra, then D is isomorphic to $\mathrm{C}(L) = \mathrm{K}(L)$ for some Stone frame L (see, e.g., [Ban89, p. 258]). Again, let X be the Priestley space of L and Y the Stone space of D. In this case, the upper topology τ_Y and its patch topology π_Y coincide. Therefore, identifying $Y = \operatorname{loc} X$, we have $\pi_Y = \tau_Y = \tau_{\operatorname{loc} X}$ (see above). Moreover, since each Stone frame is a compact regular frame, we have $\operatorname{loc} X = \min X$ (see Lemma 10.15(5)). Thus, $\operatorname{loc} X$ coincides with the set of isolated points of X, implying that $\pi_{\operatorname{loc} X}$ is discrete. This demonstrates that the restrictions of π_X and τ_X to $\operatorname{loc} X$ are distinct, showing that the operations of taking the patch topology and subspace topology may not commute.

We conclude the chapter with several remarks clarifying the role of kernels in our framework. Kernels capture essential properties of frames and provide a unifying perspective on

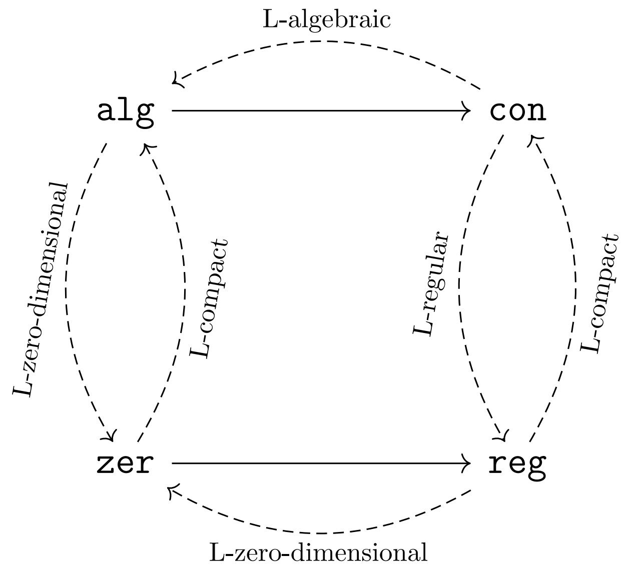

Figure 13: Relationships between kernels.

the duality results presented.

Remark 14.15. The strength of different frame properties can be understood through the way their associated kernels relate. For example, $\mathtt{alg} \leq \mathtt{con}$ (see Lemma 12.4(1)) shows that every algebraic frame is continuous. Similarly, $\mathtt{zer} \leq \mathtt{reg}$ (see Lemma 14.3(2)) implies that every zero-dimensional frame is regular. In addition, we saw that L-compactness yields $\mathtt{reg} \leq \mathtt{con}$ (see Lemma 10.14(1)), while L-regularity implies $\mathtt{con} \leq \mathtt{reg}$ (see Lemma 10.14(2)). Similarly, it can be shown that L-compactness ensures $\mathtt{zer} \leq \mathtt{alg}$, while L-zero-dimensionality implies $\mathtt{alg} \leq \mathtt{zer}$. As a consequence, for Stone L-spaces, con, reg, alg, and zer coincide (see Theorem 14.6(2)). Fig. 13 depicts the relationships between different kernels and the conditions that force them to relate to each other. In the diagram, $f \longrightarrow g$ stands for $f \leq g$, and $f \overset{\square}{\dashrightarrow} g$ for $f \leq g$ provided X is $\square$.

Remark 14.16. Given a frame L and its Priestley space X, we have:

- L is continuous iff $\mathtt{con}$ is representative (see Theorem 8.6).
- L is algebraic iff $\mathtt{alg}$ is representative (see Theorem 12.3).
- L is regular iff $\mathtt{reg}$ is representative (see Theorem 10.8(1)).
- L is zero-dimensional iff $\mathtt{zer}$ is representative (see Theorem 14.10).

We next outline how spatiality can also be characterized in terms of a kernel. Let X be an L-space. Define a map $\mathtt{spa}\colon \mathrm{ClopUp}(X) \to \mathrm{OpUp}(X)$ by

$$\mathtt{spa}\, U = \bigcup \{V \in \mathrm{ClopUp}(X) \mid V \subseteq \mathrm{cl}(U \cap \mathrm{loc}\, X)\}.$$

It is straightforward to verify that $\mathtt{spa}$ is a kernel. Moreover, $\mathtt{spa}$ is representative iff X is L-spatial. To see this, suppose $\mathtt{spa}$ is representative and $U \in \mathrm{ClopUp}(X)$. Clearly, $\mathtt{spa}\, U \subseteq \mathrm{cl}(U \cap \mathrm{loc}\, X)$, so $U = \mathrm{cl}\, \mathtt{spa}\, U \subseteq \mathrm{cl}(U \cap \mathrm{loc}\, X)$ since $\mathtt{spa}$ is representative. Therefore, $U = \mathrm{cl}(U \cap \mathrm{loc}\, X)$, so X is L-spatial by Theorem 4.4. Conversely, if X is L-spatial, then $U \subseteq \mathrm{cl}(U \cap \mathrm{loc}\, X)$ since $\mathrm{loc}\, X$ is dense. Thus, $U \subseteq \mathrm{cl}\, \mathtt{spa}\, U$, showing that $\mathtt{spa}$ is representative.

Chapter V

Conclusion

In this thesis, we have explored the relationship between Priestley duality and pointfree topology, focusing on the structure of frames via their associated Priestley spaces. This perspective allowed us to derive fundamental results, including the Hofmann–Mislove Theorem, as well as key dualities such as Hofmann–Lawson, Isbell, and Stone. In this chapter, we summarize these findings by presenting useful tables and diagrams that illustrate the relationships between the various categories considered throughout the thesis. Finally, we outline some applications of this framework and directions for future research.

Throughout this thesis, we have introduced and studied various new categories of Priestley spaces. Table 1 provides an overview of these categories along with references to their corresponding definitions. These categories serve as a foundation for our framework and can play a crucial role in understanding the structure of frames in pointfree topology.

While some of these Priestley spaces have appeared in previous work, their categories were not rigorously described previously. For example, the Priestley spaces of compact regular frames were characterized in [PS88] (see also [BGJ16]), and those of spatial and continuous frames have been studied in [PS00] (see also [ABMZ20] for spatial frames). In contrast, other categories listed in Table 1 are completely new and have not been explicitly considered before. Although we have not treated them as a separate category, we have described the Priestley spaces of zero-dimensional frames (see [BGJ16] for an earlier work in this direction).

Fig. 14 illustrates the hierarchical structure of the categories of Priestley spaces intro-

Category	**Objects** (Definition)	**Morphisms** (Definition)
Pries	Priestley spaces (2.1)	Priestley morphisms (2.3)
LPries	L-spaces (3.1)	L-morphisms (3.1)
SLPries	spatial L-spaces (4.5)	L-morphisms
KLPries	compact L-spaces (5.10)	L-morphisms
KSLPries	compact spatial L-spaces	L-morphisms
ConLPries	continuous L-spaces (8.8)	proper L-morphisms (8.8)
StCLPries	stably continuous L-spaces (9.2)	proper L-morphisms
StKLPries	stably compact L-spaces (9.2)	proper L-morphisms
KRLPries	compact regular L-spaces (10.7)	L-morphisms
AlgLPries	algebraic L-spaces (12.1)	coherent L-morphisms (12.6)
AriLPries	arithmetic L-spaces (13.1)	coherent L-morphisms
CohLPries	coherent L-spaces (13.8)	coherent L-morphisms
StoneLPries	Stone L-spaces (14.1)	L-morphisms

Table 1: Categories of Priestley spaces

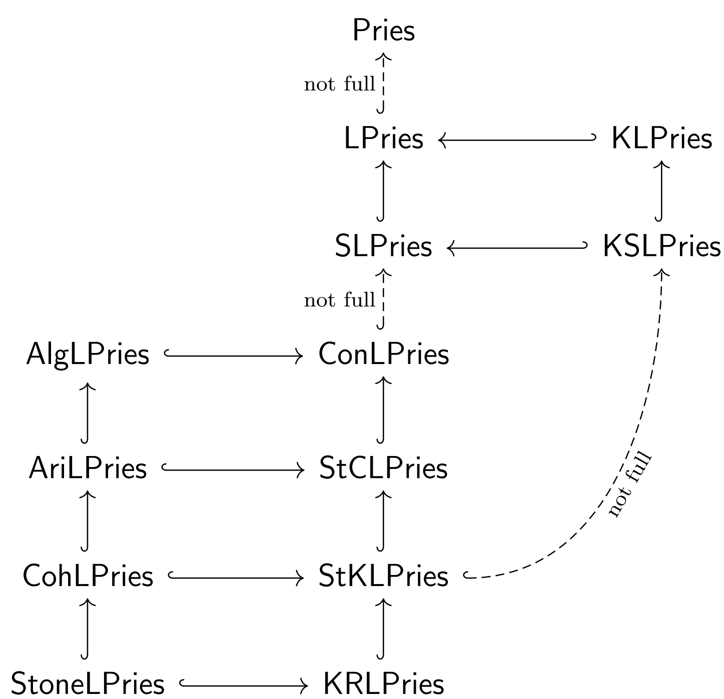

Figure 14: Relationships between categories of Priestley spaces.

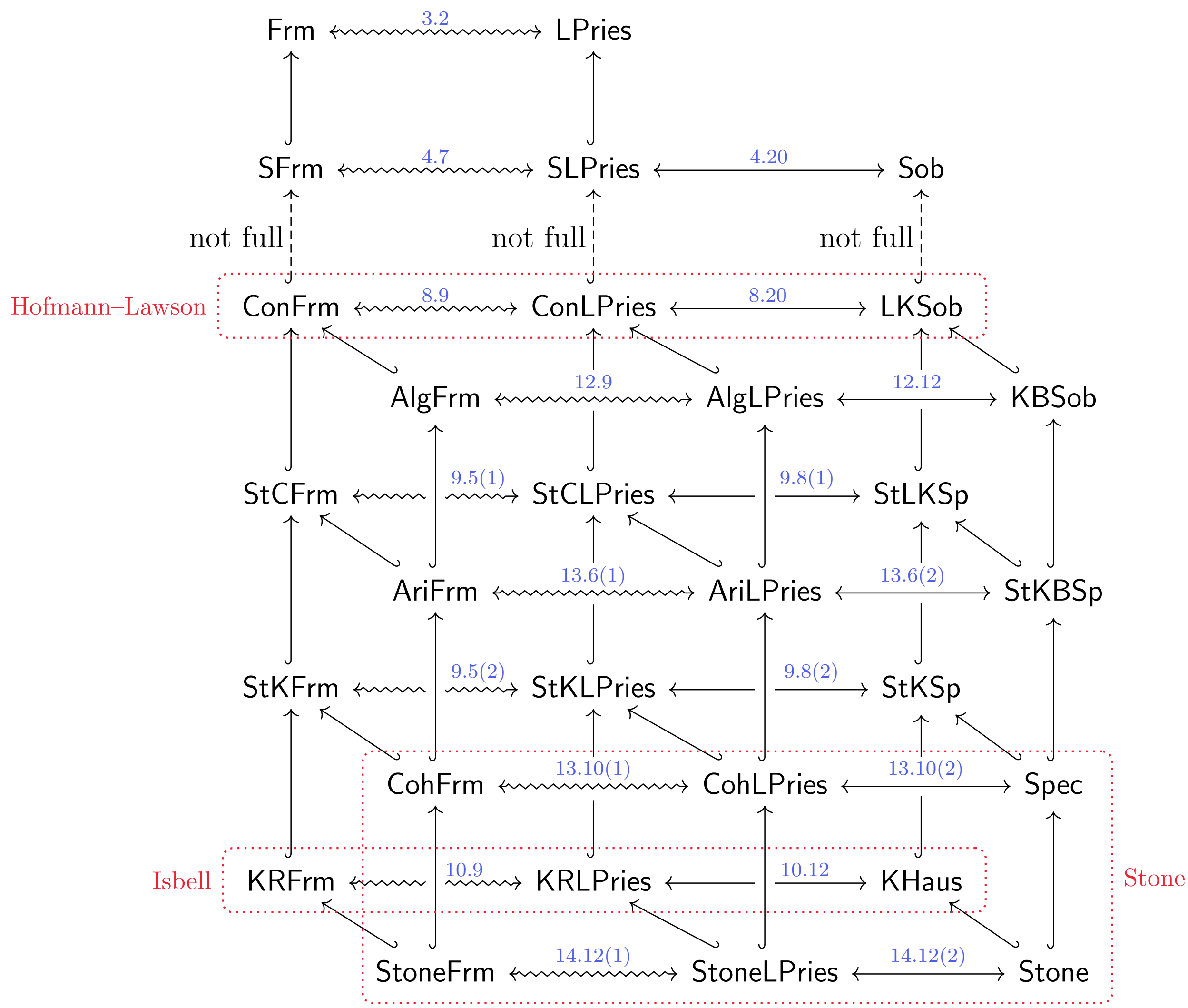

Figure 15: Relationships among categories of sober spaces, spatial L-spaces, and spatial frames.

duced in this thesis. In the diagram, $\mathsf{A} \hookrightarrow \mathsf{B}$ denotes that A is a full subcategory of B, while $\mathsf{A} \overset{\text{not full}}{\dashrightarrow} \mathsf{B}$ indicates that A is a subcategory of B but not a full subcategory.

Having introduced the new categories of Priestley spaces, we now turn to their role in providing proofs of dualities in pointfree topology. Fig. 15 illustrates these dualities, showing equivalences between categories of Priestley spaces, frames, and topological spaces. The diagram captures the key duality results of this thesis. In particular, the Hofmann–Lawson and Isbell dualities, which were the focus of Chapter III, and the Stone dualities, which were discussed in Chapter IV, are highlighted. These equivalences underscore the broader usefulness of the framework developed in this thesis. The notation follows previous diagrams, with equivalences and dual equivalences explicitly indicated.

To complement Fig. 15, we also provide a tabular overview of the relevant categories of frames and topological spaces in Tables 2 and 3, respectively. These tables list the objects and morphisms that define each category and refer to the definitions where they can be found.

Priestley duality provides a powerful framework for studying frames and their associated topological spaces. Beyond its role in duality theory, it has proven to be a useful tool in several areas of pointfree topology and lattice theory, as mentioned in the introduction. We conclude the thesis by highlighting some key applications, as well as new ideas arising from this work.

Nuclei, sublocales, and the assembly frame

Nuclei, which correspond to sublocales in pointfree topology, have been characterized via Priestley duality (see [BG07]). While this perspective has been useful in understanding the assembly frame (see, e.g., [ABMZ20, ABMZ21]) and studying certain nuclei (see, e.g., [BBM25]), much remains to be explored. An open question concerns the Priestley space of the assembly frame (the frame of nuclei):

Given an L*-space X, describe the Priestley space of the assembly frame (intrinsically as a construct of X).*

A clearer understanding of this could shed light on Isbell's famous open problem (see [Isb72, Isb91]), which asks to characterize assembly frames intrinsically.

Category	Objects (Definition)	Morphisms (Definition)
Frm	frames (1.1)	frame homomorphisms (1.4)
SFrm	spatial frames (1.3)	frame homomorphisms
KFrm	compact frames (5.5)	frame homomorphisms
KSFrm	compact spatial frames	frame homomorphisms
ConFrm	continuous frames (7.1)	proper frame homomorphisms (7.2)
StCFrm	stably continuous frames (7.3)	proper frame homomorphisms
StKFrm	stably compact frames (7.3)	proper frame homomorphisms
KRFrm	compact regular frames (7.3)	frame homomorphisms
AlgFrm	algebraic frames (11.1)	coherent frame homomorphisms (11.1)
AriFrm	arithmetic frames (11.3)	coherent frame homomorphisms
CohFrm	coherent frames (11.5)	coherent frame homomorphisms
StoneFrm	Stone frames (11.6)	frame homomorphisms

Table 2: Categories of frames

Category	Objects (Definition)	Morphisms (Definition)
Top	topological spaces	continuous maps
Sob	sober spaces (1.8)	continuous maps
KSob	compact sober spaces	continuous maps
LKSob	locally compact sober spaces (7.4)	proper continuous maps (7.4)
StLKSp	stably locally compact spaces (7.5)	proper continuous maps
StKSp	stably compact spaces (7.5)	proper continuous maps
KHaus	compact Hausdorff spaces (7.5)	continuous maps
KBSob	compactly based sober spaces (11.8)	coherent continuous maps (11.8)
StKBSp	stably compactly based spaces (11.8)	coherent continuous maps
Spec	spectral spaces (2.12)	coherent continuous maps
Stone	Stone spaces (2.6)	continuous maps

Table 3: Categories of topological spaces

Finding counterexamples via duality

In frame theory and certain generalizations, such as MT-algebras (see [BR23, BR25]) and Raney extensions (see [Sua24]), Priestley duality offers an effective method for constructing counterexamples. Working in the dual order-topological setting often allows for simpler and more intuitive ways of constructing counterexamples, which would be difficult to produce algebraically. This approach has the potential to resolve open problems in these settings by systematically translating algebraic conditions into topological ones. A concrete application of this idea was demonstrated in [BBM25] to construct a counterexample that resolved an open problem posed in [Bha19].

Studying spectra of frames through Priestley duality

The various spectra of frames can often be understood through their associated Priestley spaces. In this thesis, we observed that the point spectrum (pt) of a frame naturally appears as the localic part of its Priestley space. This provides a new way to analyze prime elements, maximal and minimal primes, and other spectral structures in a dual setting. For example, the counterexample produced in [BBM25] was obtained by characterizing the spectrum of maximal d-elements in terms of L-spaces. Building on these methods, additional natural spectra and their dual descriptions will be explored, extending the scope of Priestley duality in pointfree topology.

Object	**Collection**	**Description**	**Page**
Priestley/L-space X	$\mathrm{ClopUp}(X)$	clopen upsets	12
	$\mathrm{ClopDn}(X)$	clopen downsets	13
	$\min X$	minimal points	14
	$\max X$	maximal points	14
	$\mathrm{ClUp}(X)$	closed upsets	14
	$\mathrm{OpUp}(X)$	open upsets	16
	$\mathrm{loc}\, X$	localic points	21
	$\mathrm{SUp}(X)$	Scott upsets	32
	$\mathrm{ClopSUp}(X)$	clopen Scott upsets	88
	$\mathrm{ClopBi}(X)$	clopen bisets	99
Lattice/Frame L	$\mathrm{pt}(L)$	completely prime filters	8
	$\mathrm{Filt}(L)$	filters	14
	$\mathrm{OFilt}(L)$	Scott-open filters	38
	$\mathrm{K}(L)$	compact elements	83
	$\mathrm{C}(L)$	complemented elements	83
Topological space Y	$\Omega(Y)$	open sets	7
	$\mathrm{KSat}(Y)$	compact saturated sets	38

Table 4: The notation used throughout the text.

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