Priestley Duality for d-Frames

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Pointfree topology

A frame (or locale) is a complete lattice whose finite meets distribute over arbitrary joins.

Frames are used to study topological spaces via their lattices of open sets. Indeed, the lattice of open sets of each topological space is a frame.

This allows us to study spaces without points, hence the name pointfree topology.

Nuclei

The pointfree analogue of a subspace is a sublocale.

A subset of a frame is a sublocale if the left adjoint of the inclusion map is a nucleus: a closure operator that commutes with finite meets.

For each nucleus $j : L \to L$, the set $jL = \{a \in L : j(a) = a\}$ is the corresponding sublocale.

Algebraic lattices

A complete lattice is algebraic if each element is the join of the compact elements below it.

As the name suggests, algebraic lattices have their origin in algebra:

- The lattice of congruences of any algebra is algebraic.
- The lattice of subuniverses of any algebra is algebraic.

If an algebraic lattice is distributive, it is a frame.

Algebraic frames

There are several important categories of algebraic frames.

An arithmetic frame is an algebraic frame where binary meets of compact elements are compact.

A coherent frame is an arithmetic frame that in addition is compact.

A frame homomorphism is coherent if it maps compact elements to compact elements.

We consider the following categories of algebraic frames.

AlgFrm – algebraic frames and coherent frame homomorphisms

AriFrm – full subcategory of arithmetic frames

CohFrm – full subcategory of coherent frames

Compactly based spaces

There is a well-known dual equivalence between the categories Sob of sober spaces and SFrm of spatial frames.

This restricts to a dual equivalence between AlgFrm and the category KBSob of compactly based sober spaces and coherent continuous maps.

- A space is compactly based if it has a basis of compact sets.
- A continuous map is coherent if the inverse image of a compact open set is compact.

Theorem (Hofmann and Keimel, 1972)

AlgFrm *and* KBSob *are dually equivalent.*

Restricting further to the categories of arithmetic and coherent frames yields the following dualities

$$
\begin{array}{ccccc}
\mathsf{AlgFrm} & \geqslant & \mathsf{AriFrm} & \geqslant & \mathsf{CohFrm} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathsf{KBSob} & \geqslant & \mathsf{SKBSob} & \geqslant & \mathsf{Spec}
\end{array}
$$

where we have the following full subcategories of KBSob:

SKBSob – stably compactly based sober spaces
(binary intersection of compact opens is compact)

Spec – spectral spaces
(stably compactly based + compact)

d-nucleus

Let L be an arithmetic frame. Define $d : L \to L$ by

$$d(a) = \bigvee\{k^{**} \mid k \text{ is compact and } k \leqslant a\}.$$

Lemma (Martinez and Zenk, 2003)

1. *d is a nucleus.*
2. *$d(0) = 0$, i.e., d is dense.*
3. *$d(a) = \bigvee\{d(k) \mid k \in K(L), k \leqslant a\}$, i.e., d is inductive.*
4. *d is the greatest dense and inductive nucleus on L.*

Motivation

Martinez and Zenk introduced the d-nucleus as a frame-theoretic tool to study d-ideals.

The study of d-ideals was started in the 1970s/80s by several authors, e.g., Bernau; Luxemburg; Bondarev; Bigard, Keimel, and Wolfenstein; Huijsmans and de Pagter.

d-ideals have become an important object of study in the setting of ℓ-groups.

In this talk, we use the machinery of Priestley spaces to provide a new perspective on the study of the d-nucleus. We hope that this will shed new light on some open problems in the area.

Priestley duality

A Priestley space is a partially ordered compact space with the property that if $x \nleq y$ then x can be separated from y by a clopen upset.

Theorem (Priestley, 1970)

DLat *and* Pries *are dually equivalent.*

where

DLat is the category of bounded distributive lattices and their homomorphisms

Pries is the category of Priestley spaces and order-preserving continuous maps

Pultr-Sichler duality

Since frames are bounded distributive lattices, there is a class of Priestley spaces corresponding to frames.

- An Esakia space is a Priestley space such the downset of each clopen is clopen.
- An Esakia space is extremally order-disconnected if the closure of each open upset is clopen.
- An L-space (localic space) is an extremally order-disconnected Esakia space.
- An L-morphism is an order-preserving continuous map f such that $\mathsf{cl}\, f^{-1}(U) = f^{-1}(\mathsf{cl}\, U)$ for each open upset U.

Let LPries be the category of L-spaces and L-morphisms.

Theorem (Pultr and Sichler, 1988)

Frm *and* LPries *are dually equivalent.*

Priestley duality for spatial frames

Definition

Let X be an L-space.

- The localic part of X is $Y := \{y \in X \mid {\downarrow}y \text{ is } \textit{clopen}\}$.
- A closed upset $F \subseteq X$ is a Scott upset if $\min F \subseteq Y$.
- X is an SL-space if every clopen upset is the closure of the Scott upsets it contains. (Equivalently, Y is dense in X)

Let SLPries be the full subcategory of LPries consisting of SL-spaces.

Theorem

SFrm *and* SLPries *are dually equivalent.*

Priestley duality for algebraic frames

Definition

Let X, X' be L-spaces.

- Let $\mathsf{ClopSUp}(X)$ be the collection of clopen Scott upsets of X.
- For a clopen upset $U \subseteq X$, the core of U is

$$\operatorname{core} U = \bigcup \{V \in \mathsf{ClopSUp}(X) \mid V \subseteq U\}.$$

- X is algebraic if $\operatorname{core} U$ is dense in U for each clopen upset $U \subseteq X$.
- An L-morphism $f : X \to X'$ is coherent if $f^{-1}(\operatorname{core} U) \subseteq \operatorname{core} f^{-1}(U)$.

Let AlgLPries be the category of algebraic L-spaces and coherent L-morphisms.

Theorem

AlgFrm *and* AlgLPries *are dually equivalent.*

Priestley duality for arithmetic/coherent frames

Definition

- An arithmetic L-space is an algebraic L-space such that $\operatorname{core} U \cap \operatorname{core} V = \operatorname{core}(U \cap V)$ for all clopen upsets $U, V \subseteq X$.
- An L-space X is L-compact if $\min X \subseteq Y$.
- A coherent L-space is an arithmetic L-space that is also L-compact.

Let AriLPries and CohLPries be the corresponding full subcategories of AlgLPries.

Theorem

1. AriFrm *and* AriLPries *are dually equivalent.*
2. CohFrm *and* CohLPries *are dually equivalent.*

Nuclear subsets

Let L be a frame and X its Priestley space.

Definition

A closed set $N \subseteq X$ is nuclear if $\downarrow(U \cap N)$ is clopen for each clopen $U \subseteq X$.

Theorem (Bezhanishvili and Ghilardi, 2007)

There is a one-to-one correspondence between nuclei of L and nuclear subsets of X.

If we write N_j for the nuclear subset corresponding to a nucleus $j : L \to L$, then N_j is the Priestley space of jL.

N_d and Y_d

Let L be a frame, X its Priestley space, Y the localic part of X, and $j \in N(L)$.

Lemma

1. *$Y_j := N_j \cap Y$ is the localic part of N_j.*
2. $N_d = \mathsf{cl}(Y_d)$.
3. $\max Y \subseteq Y_d$ (*and this inclusion can be strict*).

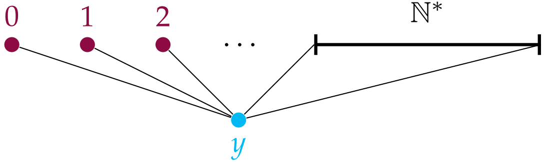
Theorem

For $y \in Y$, the following are equivalent.

1. $y \in Y_d$.
2. $\max{\uparrow}y \subseteq V$ *implies* $y \in V$ *for all* $V \in \mathsf{ClopSUp}(X)$.
3. *there is* $x \in \max X$ *such that* $y = \max({\downarrow}x \cap Y)$.

Example $\max Y \subsetneq Y_d$

Let $X =$



- $Y = \mathbb{N} \cup \{y\}$
- $\max Y = \mathbb{N}$
- $Y_d = Y$

Maximal d-elements

The collection of maximal d-ideals equipped with the hull-kernel topology has been studied extensively.

The frame-theoretic analogue of this is the space $\max dL$ of maximal d-elements (see Bhattacharjee, 2019).

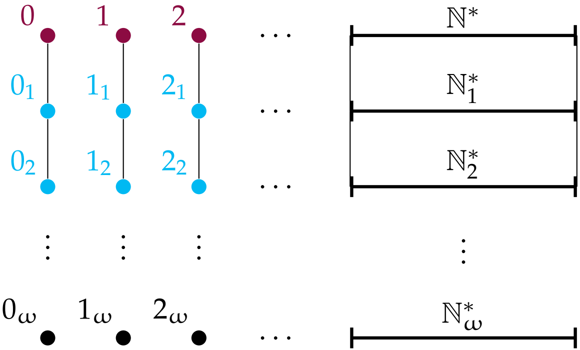
It is an open question whether $\max dL$ is always Hausdorff.

We hope our machinery will be useful to answer this question.

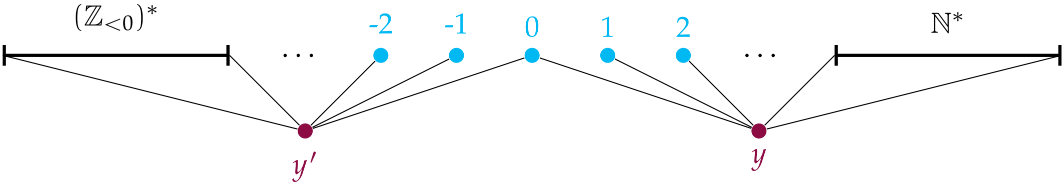
Lemma

$\min Y_d$ *is homeomorphic to* $\max dL$.

- $\Upsilon = \mathbb{N} = \Upsilon_d = \min \Upsilon_d$



- $Y = \bigcup_{n<\omega} \mathbb{N}_n$
- $Y_d = \mathbb{N}_0 = \min Y_d$



- $Y = \mathbb{Z} \cup \{y, y'\} = Y_d$
- $\min Y_d = \{y, y'\}$

What we know about $\min Y_d$

Let X be an arithmetic L-space.

Lemma

1. $\min Y_d$ *is a locally compact* T_1*-space.*
2. $\min Y_d$ *is compact iff there is* $U \in \mathsf{ClopSUp}(X)$ *with* $\max X \subseteq V$.
3. $\min Y_d$ *is Hausdorff iff* $\min Y_d$ *is sober.*

Lemma

Let X *be the Priestley space of a regular arithmetic frame. Then* $Y \subseteq \min X$*, and hence* $\min Y_d = Y = \max Y$ *is locally Stone (locally compact, Hausdorff, and zero-dimensional).*

Thank you!

¡Gracias!