The maximal spectrum of d-elements is not always Hausdorff

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d-ideals

The concept of d-ideals has been extensively studied in the Riesz space literature (see, e.g., Luxemburg and Zaanen, 1971).

Huijsmans and de Pagter (1983) studied the topological properties of the spectrum of maximal d-ideals of archimedean Riesz spaces with a weak order unit.

They showed that this spectrum is a compact Hausdorff space.

Martinez and Zenk (2003) observed that these considerations fall under the umbrella of studying the d-nucleus on arithmetic frames.

Arithmetic frames

A frame is a complete lattice L satisfying $a \wedge \bigvee b_i = \bigvee(a \wedge b_i)$ for all $a, b_i \in L$.

A frame is algebraic if it satisfies $a = \bigvee\{b \in K(L) \mid b \leq a\}$ for all $a \in L$, where $K(L)$ is the collection of compact elements of L.

An arithmetic frame (or M-frame) is an algebraic frame where $a \wedge b \in K(L)$ for all $a, b \in K(L)$.

Note that arithmetic frames are a not necessarily compact, i.e., 1 may not be in $K(L)$. Thus, arithmetic frames are a generalization of coherent frames, which are studied extensively in the literature.

The d-nucleus

Let L be an arithmetic frame.

Definition (Martinez and Zenk, 2003)

The d-nucleus is defined as

$$da = \bigvee\{b^{**} \mid b \in K(L) \text{ and } b \leq a\}$$

for all $a \in L$.

where a^* stands for the pseudocomplement of a.

We denote the sublocale of fixpoints of d by dL.

Definition (Bhattacharjee, 2019)

The spectrum max(dL) is the collection of maximal elements of dL equipped with the topology $\{\max(dL) \setminus {\uparrow}a \mid a \in L\}$.

Huijsmans and de Pagter studied archimedean Riesz spaces with a weak order unit.

The frame theoretic analogue is:

Definition (Bhattacharjee, 2019)

A compact dense element of L is called a unit.

where we recall an element $a \in L$ is dense if $a^* = 0$.

$\max(dL)$ is compact

Theorem (Bhattacharjee, 2019)

Let L be an arithmetic frame with a unit. Then $\max(dL)$ *is a compact T_1-space.*

The question of whether $\max(dL)$ is Hausdorff was left open.

The aim of this is talk is to resolve this question in the negative.

Our main machinery is Priestley duality for frames.

Priestley duality

A Priestley space is a partially ordered compact space $(X, \leq)$ such that $x \nleq y$ implies that there exists a clopen upset containing x and missing y.

Theorem (Priestley, 1970)

The category of bounded distributive lattices and the category of Priestley spaces are dually equivalent.

Priestley duality for frames

Priestley duality was restricted to frames by Pultr & Sichler.

Definition

An L-space (localic space) is a Priestley space such that the closure of each open upset is an open upset.

Theorem (Pultr-Sichler, 1988)

The category of frames and the category of L-spaces are dually equivalent.

Priestley duality for spatial frames

Arithmetic frames are spatial—they are isomorphic to the frames of opens of some topological space.

In an L-space X, this corresponds to the density of a special subset.

Definition

Let X be an L-space.

1. The localic part of X is $Y = \{y \in X \mid {\downarrow}y \text{ is open}\}$ and points of Y are called localic.
2. X is an SL-space if Y is dense in X.

Theorem (Pultr-Sichler, 1988)

The category of spatial frames is dually equivalent to the category of SL-spaces.

Arithmetic L-spaces and the Priestley space of dL

We further restricted Pultr-Sichler duality to the category of arithmetic frames. The corresponding Priestley spaces are called arithmetic L-spaces (and are characterized by an appropriate density condition).

Let X_d be the Priestley space of dL, and let Y_d be its localic part. We can realize X_d as a special closed set of the Priestley space X of L.

Moreover, we have the following.

Lemma

1. $Y_d = X_d \cap Y$.
2. $y \in Y_d$ *iff* y *is the greatest localic point below a maximal point of* X.

$\min(Y_d) \cong \max(dL)$

Let $\min(Y_d)$ be the collection of minimal points of Y_d.

Lemma

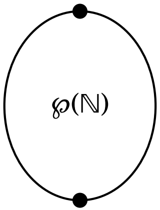
$\min(Y_d)$ *is in bijection with* $\max(dL)$.

By topologizing $\min(Y_d)$ with $\{U \cap \min(Y_d) \mid U \text{ is a clopen upset of } X\}$ we obtain the following theorem.

Theorem

$\max(dL)$ *is homeomorphic to* $\min(Y_d)$.

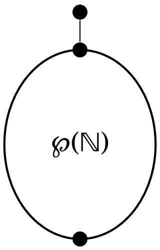
A simple example

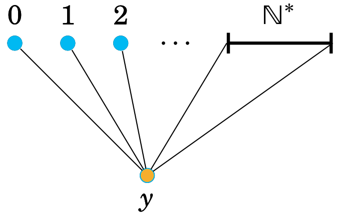


Let $L = \wp(\mathbb{N})$ be the powerset of the natural numbers. The Priestley space X of L is the Stone-Čech compactification of $\mathbb{N}$.

- $Y = \mathbb{N}$.
- $Y_d = \mathbb{N}$.
- $\min Y_d = \mathbb{N}$ is discrete.

A simpler example





Let L be the powerset of the natural numbers with a new top element. The Priestley space X of L is the Stone-Čech compactification of $\mathbb{N}$ with a new point y below.

- $Y = \mathbb{N} \cup \{y\}$.
- $Y_d = \mathbb{N} \cup \{y\}$.
- $\min Y_d = \{y\}$ is a singleton space.

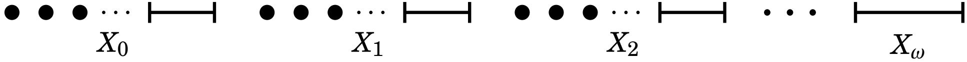
Constructing a non-Hausdorff $\min(Y_d)$

We now produce an example of an arithmetic L-space X such that $\min(Y_d)$ is not Hausdorff.

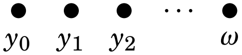
Take the Stone-Čech compactification of the natural numbers

$\beta\mathbb{N} =$ 

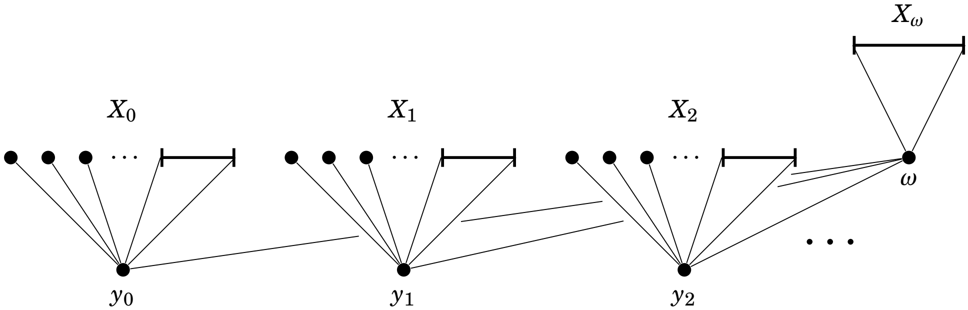
and partition it $\beta\mathbb{N} = (\bigcup X_i) \cup X_\omega$ into countably infinitely many copies X_i of $\beta\mathbb{N}$ and a subset $X_\omega \subseteq \mathbb{N}^*$.

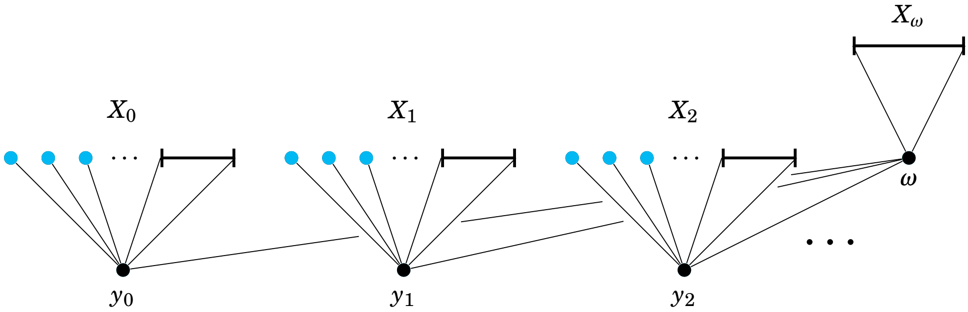


Then take the disjoint union with the one point compactification of the natural numbers



and equip the space with the following order





- X is the L-space of an arithmetic frame with a unit.
- $Y = Y_d = \mathbb{N} \cup \{y_0, y_1, \dots\} \cup \{\omega\}$.
- $\min(Y_d) = \{y_0, y_1, \dots\}$.
- $\min(Y_d)$ is homeomorphic to the natural numbers with cofinite topology.
- $\min(Y_d)$ is not Hausdorff.

Conclusion

The example shows that there exist arithmetic frames L with a unit such that $\max(dL)$ is not Hausdorff.

Some open questions:

- When is $\min(Y_d)$ Hausdorff?
- When is $\min(Y_d)$ locally compact? In particular, which subsets of X intersect down to compact sets of $\min(Y_d)$?

Thank you for your attention!