Canonical Formulas for the Lax Logic

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Lax Logic

Lax Logic is an intuitionistic modal logic that naturally appears in many fields of mathematics and computer science.

It features a modal operator $\bigcirc$ that has properties of classical necessitation and possibility.

This has lead to several interesting interpretations for $\bigcirc\phi$:

- Goldblatt (1981): "ϕ is locally true."
- Fairtlough and Mendler (1997): "ϕ holds for some constraint."
- Benton, Bierman, and Paiva (1998): "a computation of type ϕ" or "the possibility of a value of type ϕ."

Most of the readings appear to be existential but nevertheless in intuitionistic normal modal logic this modality is formalized with $\Box$.

Lax Logic Formally

We will work with the following language:

$$\mathcal{L} \ni \phi ::= p \mid \bot \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi \mid \bigcirc\phi$$

Definition

A set $L \subseteq \mathcal{L}$ is a lax logic iff it contains the axioms of the intuitionistic propositional calculus (IPC), the axioms

$$p \to \bigcirc p$$
$$\bigcirc\bigcirc p \to \bigcirc p$$
$$(\bigcirc p \wedge \bigcirc q) \to \bigcirc(p \wedge q)$$

and is closed under the rules of modus ponens, uniform substitution, and the rule of regularization: from $\phi \to \psi$ infer $\bigcirc\phi \to \bigcirc\psi$.

The first step towards studying lax modalities may have been by Dedekind (1888). He found that all closure operators $f : X \to X$ satisfy

$$x \leq fy \Leftrightarrow fx \leq fy.$$

Macnab (1981) later characterized lax modalities on Heyting algebras by functions $\bigcirc : A \to A$ that satisfy

$$a \to \bigcirc b = \bigcirc a \to \bigcirc b.$$

Indeed, lax modalities are multiplicative closure operators on Heyting algebras.

Lax logics are everywhere

- Proof theory: Curry (1952), Aczel (2001), van den Berg (2019)
- Pointless topology: Dowker and Papert (1966), Simmons (1978).
- Topos theory: Lawvere (1970), Fourman and Scott (1979), Goldblatt (1981).
- Formal hardware verification: Fairtlough and Mendler (1997).
- Computational lambda calculus: Benton, Bierman, and Paiva (1998).
- Subframe logics: G. Bezhanishvili and Ghilardi (2007), G. Bezhanishvili, N. Bezhanishvili, and Ilin (2019).

Names used for lax modalities include: multiplicative closure operator, nucleus, local operator, modal operator, and j-operator.

Nuclear algebras

Heyting algebras with lax modalities are known as nuclear algebras and they give the algebraic semantics for lax logic.

Theorem (Goldblatt, 1981)

Nuclear algebras provide complete semantics for the lax logic.

The name nuclear is derived from nucleus which is a common name for lax modalities in lattice and topos theory.

Goldblatt called them local operators corresponding with his local reading of the modality.

Relational semantics

Goldblatt (1981) also introduced complete Kripke semantics for the lax logic.

A Goldblatt frame is a tuple $(X, \leq, R)$ such that $\leq$ is a partial order and:

$$\leq \circ R \subseteq R, \qquad R \subseteq \leq, \qquad R \subseteq R^2.$$

The Kripke semantics ($\Vdash$) is determined by the expected clauses:

$$
\begin{aligned}
X, x \Vdash p &\iff x \in v(p) \\
X, x \Vdash \bot &\iff \text{never} \\
X, x \Vdash \phi \wedge \psi &\iff X, x \Vdash \phi \text{ and } X, x \Vdash \psi \\
X, x \Vdash \phi \vee \psi &\iff X, x \Vdash \phi \text{ or } X, x \Vdash \psi \\
X, x \Vdash \phi \to \psi &\iff x \leq y \text{ and } X, y \Vdash \phi \text{ implies } y \Vdash \psi \\
X, x \Vdash \bigcirc\phi &\iff xRy \text{ implies } X, y \Vdash \phi
\end{aligned}
$$

The prevalence of lax logic in the literature has also lead to other Kripke style semantics.

Fairtlough and Mendler (1997) used frames with fallible worlds to give the clause for $\bigcirc$ an existential and universal character:

$$X, x \Vdash \bigcirc\phi \iff \forall y : x \leq y \text{ implies } \exists z : yRz \text{ and } X, z \Vdash \phi$$

G. Bezhanishvili, N. Bezhanishvili, and Ilin (2019) introduced S-frame semantics. Instead of interpreting the modality with a binary relation $R \subseteq X^2$ they make use of a subset $S \subseteq X$:

$$X, x \Vdash \bigcirc\phi \iff \forall y \in S : x \leq y \text{ implies } X, y \Vdash \phi$$

Besides, there is no accepted standard for the confluence condition $\leq \circ R \subseteq R$ in intuitionistic modal logic. The most common seems to be $\leq \circ R \circ \leq = R$, e.g. Sotirov (1984) or Wolter and Zakharyaschev (1999).

All these relational semantics are complete for the least lax logic PLL, but this is not the case for all lax logics.

However, completeness coincides in the finite case:

Theorem

Let L be a lax logic. The following are equivalent.

1. L is complete with respect to finite Goldblatt frames.
2. L is complete with respect to finite FM-frames.
3. L is complete with respect to finite S-frames.
4. L is complete with respect to finite IK-frames.

Ultimately, all can be considered as Kripke-completeness in the context of lax logics. We will use IK-frames.

Definition

A lax logic is Kripke-complete iff it is complete with respect to IK-frames.

Translations

IPC-definable lax modalities provide a Gödel-Gentzen style translation from IPC into itself, e.g. the $\neg\neg$-translation.

More generally we can study such translations from the intuitionistic language into $\mathcal{L}$:

- The inner space translation $(\_)^\circ$ is defined as:

$$\begin{aligned} p^\circ &= \bigcirc p \\ \bot^\circ &= \bigcirc \bot \\ (\phi \wedge \psi)^\circ &= \phi^\circ \wedge \psi^\circ \\ (\phi \vee \psi)^\circ &= \bigcirc(\phi^\circ \vee \psi^\circ) \\ (\phi \to \psi)^\circ &= \phi^\circ \to \psi^\circ \end{aligned}$$

- The outer space translation $(\_)^\bullet$ is the identity.

We can extend these translations to logics, e.g. if L is an intermediate logic then we define $L^\circ := \mathsf{PLL} \oplus \{\phi^\circ \mid \phi \in L\}$ and $L^\bullet := \mathsf{PLL} \oplus L$.

The inner and outer space translation were introduced by G. Bezhanishvili, N. Bezhanishvili, and Ilin (2019) to characterize subframizations of intermediate logics.

The names are inspired by the determination of validity in the relevant spaces. An S-space $\mathcal{X} = (X, S)$ is a pair of Esakia spaces such that S is a (descriptive) subframe of X.

- $\mathcal{X}$ is an L°-space iff the inner space S is an L-space
- $\mathcal{X}$ is an $L^\bullet$-space iff the outer space X is an L-space

The aim

Today we will investigate which logical properties are preserved through the outer space translation.

In particular, we will focus on Kripke-completeness.

Conjecture

If L is Kripke-complete then $L^\bullet$ is Kripke-complete.

Other considerations:

- finite model property (fmp)
- tabularity
- decidability

Canonical formulas

Zakharyaschev ('80s-'90s) introduced canonical formulas as a method to uniformly axiomatize all intermediate (and transitive modal logics).

This method provided a lot of structure in the study of intermediate logics:

- They describe logics with geometric refutation patterns.
- Simple instances of canonical formulas characterize subframe logics.
- They give insight into the relation of intermediate logics and their modal companions.
  - For example, Zakharyaschev used them to obtain a positive answer for the Dummett-Lemmon conjecture.

The method of canonical formulas

Essentially, the method of canonical formulas is a two-step procedure:

1. Characterize every formula ϕ with a finite number of refutation patterns.
   - This is a finite collection of counter-models $A_1, \ldots, A_n$ with some parameters $D_1, \ldots, D_n$.
   - We can use both algebras and frames.
2. Encode the refutation patterns into formulas: $\alpha(A_i, D_i)$.
   - We have to make sure they are semantically equivalent, i.e., ϕ is valid in a frame iff $\bigwedge_i \alpha(A_i, D_i)$ is.

Consequently, they will axiomatise every logic.

Limitations of canonical formulas

Zakharyaschev's approach to canonical formulas relies on the dual structure of the finitely generated algebras of intermediate and transitive modal logics.

Consequently, it has only been applied to intermediate and transitive modal logics.

G. Bezhanishvili and N. Bezhanishvili ('00s-'10s) have provided an uniform algebraic approach to the method relying on local finiteness of relevant algebras.

The intermediate case

Finding counter-models

We will construct canonical formulas using the $\vee$-free reduct of Heyting algebras $\{\wedge, \vee, \rightarrow, \bot\}$.

Theorem (Diego, 1966)

The variety of bounded implicative semilattices ($\{\wedge, \rightarrow, \bot\}$-algebras) is locally finite.

Consequently, there is a bound on the number of k-generated $\{\wedge, \rightarrow, \bot\}$-algebras.

Suppose $\mathsf{IPC} \nvdash \phi$. Let $k = |\mathsf{sub}(\phi)|$.

By Diego's Theorem there are only finitely many Heyting algebras $A_1, \ldots, A_n$ that are k-generated as $\{\wedge, \to, \bot\}$-algebras and refute ϕ.

For each A_i we have a valuation v_i that witnesses the refutation of ϕ. We use these valuations to populate the parameters $D_1, \ldots, D_n$:

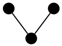
$$D_i = \{v_i(\psi), v_i(\chi) \mid \psi \vee \chi \in \mathsf{sub}(\phi)\}.$$

This gives the geometric refutation patterns for step 1.

Geometric refutation patterns

Ultimately, we will characterize logics by the geometric patterns they refute.

For example, LC – the intermediate logic generated by the class of chains is characterized by refuting the frame:



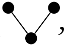
Any frame that contains a “fork” cannot refute this pattern!

Encoding refutation patterns

Let A be a finite Heyting algebra with the second largest element s. The canonical formula associated with A and $D \subseteq A^2$ is defined as

$$\alpha(A, D) = [(\bigwedge_{a,b \in A^2} p_{a \wedge b} \leftrightarrow p_a \wedge p_b) \wedge (\bigwedge_{a,b \in A^2} p_{a \to b} \leftrightarrow p_a \to p_b) \\ \wedge (\bigwedge_{a,b \in D} p_{a \vee b} \leftrightarrow p_a \vee p_b) \wedge (p_\bot \leftrightarrow \bot)] \to p_s$$

where p_a is a fresh variable for each $a \in A$.

In other words, even $\alpha($$, \emptyset)$ is a huge jumble of symbols.

Refutation criterion

Fortunately, the specific implementation does not matter. Crucial is only the refutation criterion.

$$B \twoheadrightarrow C \xhookleftarrow[\vee : D]{h} A$$

$B \nvDash \alpha(A, D)$ iff there is a homomorphic image C of B and a $\{\wedge, \to, \bot\}$-embedding $h : A \to C$ that is $\vee$-compatible over D, i.e.,

$$h(a \vee b) = ha \vee hb \qquad \text{for all } a, b \in D.$$

Intuitively, B refutes $\alpha(A, D)$ iff the full $\{\wedge, \to, \bot\}$-structure of A is present in B and the $\vee$-structure of A is present in B partially up to D.

Equivalence of formulas

Suppose $(A_1, D_1), \dots, (A_n, D_n)$ are the geometric refutation patterns for ϕ.

We need to show that $B \models \phi$ iff $B \models \bigwedge_i \alpha(A_i, D_i)$.

$(\Rightarrow)$ is a consequence of the refutation criterion and how we populated the parameter sets.

The trickier part is $(\Leftarrow)$ and we need to use an algebraic account of filtration.

Specifically, we expand a finitely generated $\{\wedge, \rightarrow, \bot\}$-algebra back to a full Heyting algebra.

We obtain a finite $\{\wedge, \to, \bot\}$-subalgebra A of B using the locally finite reduct.

$$B \twoheadrightarrow B \xhookleftarrow[\vee:D]{i} A$$

We can expand A into a Heyting algebra by defining

$$a \vee_A b := \bigwedge \{c \in A \mid c \geq a \vee_B b\}.$$

It follows by induction that $A \nvDash \phi$.

Besides, i is $\vee$-compatible over $D = \{(a, b) \mid a \vee_B b \in v[\mathsf{sub}(\phi)]\}$.

Clearly, B is a homomorphic image of itself.

Whence, $B \nvDash \alpha(A, D)$ for some geometric refutation pattern for ϕ.

Thus, we have $\mathsf{IPC} \oplus \phi = \mathsf{IPC} \oplus \alpha(A_1, D_1) \oplus \cdots \oplus \alpha(A_n, D_n)$ for every formula ϕ.

Theorem (Zakharyaschev, 1989)

All intermediate logics are axiomatized by canonical formulas.

The main ingredients of canonical formulas:

1. A locally finite reduct that can be expanded back into the full type faithfully (filtration).
2. A process to encode patterns into formulas with the right refutation criterion.

Subframes

Canonical formulas have an interesting connection to subframe logics.

Let $\mathcal{X} = (X, \leq_X)$ and $\mathcal{Y} = (Y, \leq_Y)$ be intuitionistic Kripke frames.

Definition

$\mathcal{Y}$ is a subframe of $\mathcal{X}$ iff

- $Y \subseteq X$
- $\leq_Y = \leq_X \cap Y^2$.

Note that there is a one-to-one relation between subsets of X and subframes of $\mathcal{X}$.

In other words, we can think of the subframes of a frame as its subsets.

Subframe logics

Subframes characterize a well-behaved class of intermediate logics.

Definition

A class of frames K is closed under subframes if

$$\mathcal{X} \in \mathsf{K} \text{ and } \mathcal{Y} \text{ is a subframe of } \mathcal{X} \text{ implies } \mathcal{Y} \in \mathsf{K}.$$

Definition

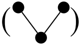
An intermediate logic is a subframe logic if it is generated by a class closed under subframes.

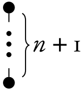
Subframe formulas

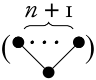
Subframe logics are exactly those intermediate logics that are axiomatized by canonical formulas of the form $\alpha(A) := \alpha(A, \emptyset)$. We call these subframe formulas.

Some examples of subframe logics:

$\mathsf{CPC} = \mathsf{IPC} \oplus \alpha($  $)$

$\mathsf{LC} = \mathsf{IPC} \oplus \alpha($  $)$

$\mathsf{BD}_n = \mathsf{IPC} \oplus \alpha($  $)$

$\mathsf{BW}_n = \mathsf{IPC} \oplus \alpha($  $)$

The lax case

Lax canonical formulas

Recall the ingredients of canonical formulas:

1. A locally finite reduct that can be expanded back into the full type faithfully. ✓

Theorem (G. Bezhanishvili, N. Bezhanishvili, Carai, Gabelaia, Ghilardi, and Jibladze, 2020)

The $\vee$-free reduct of nuclear algebras locally finite.

2. A process to encode patterns into formulas with the right refutation criterion: $B \twoheadrightarrow C \underset{\vee:D}{\hookleftarrow} A$ ✓

$$\beta(A, D) = [(\bigwedge_{a,b\in A^2} p_{a\wedge b} \leftrightarrow p_a \wedge p_b) \wedge (\bigwedge_{a,b\in A^2} p_{a\to b} \leftrightarrow p_a \to p_b)$$

$$\wedge(\bigwedge_{a,b\in D} p_{a\vee b} \leftrightarrow p_a \vee p_b) \wedge (p_\bot \leftrightarrow \bot) \wedge (\bigwedge_{a\in A} p_{\bigcirc a} \leftrightarrow \bigcirc p_a)] \to p_s$$

We call $\beta(A, D)$ a lax canonical formula. It encodes the complete $\vee$-free reduct of the algebra and $\vee$ only partially up to D.

Lax canonical formulas are a consequence of Diego's Theorem for $\vee$-free nuclear algebras.

Theorem

All lax logics are axiomatized by lax canonical formulas.

However, lax canonical formulas are not perfect...

Lax subframe logics

In the intermediate case subframe logics are axiomatized by canonical formulas of the form $\alpha(A, \emptyset)$.

Should lax canonical formulas of the form $\beta(A, \emptyset)$ axiomatize lax subframe logics?

Yes, because we want it to mirror the intermediate case.

But it is not feasible.

Finite domains

Partial Esakia morphisms are a dual representation of $\{\wedge, \to, \bot\}$-homomorphisms.

In the intermediate setting finite domains of onto partial Esakia morphisms instantiate subframes.

This means that subframe logics are closed under finite domains of onto partial Esakia morphisms (finite domain property).

However, lax logics axiomatized by canonical formulas of the form $\beta(A, \emptyset)$ do not generally satisfy this property for partial nuclear morphisms – the dual of $\{\wedge, \to, \bot, \bigcirc\}$-homomorphisms.

The problem is that we cannot define lax subframes by simply restricting the lax relation to a subset.

Lax subframes

Consider the following lax frame.

If we restrict R to $\{x, z\}$ then it is not a lax relation: $R' \nsubseteq (R')^2$.

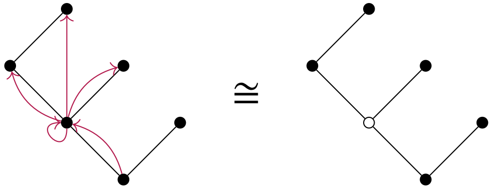
Either lax subframes do not correspond to subsets or we have to define R' differently.

So how to define lax subframes? Luckily, we have all these semantics!

S-spaces

G. Bezhanishvili and Ghilardi (2007) showed that descriptive IK-frames and S-spaces (descriptive S-frames) are in a one-to-one correspondence.

Intuitively, lax relations are determined by their reflexive points:



This forms the connection of lax logics and subframe logics studied by G. Bezhanishvili, N. Bezhanishvili, and Ilin (2019).

Steady subframes

S-frames inspire a neat definition for lax subframes.

Definition

$(Y, \leq_Y, S_Y)$ is a steady subframe of $(X, \leq_X, S_Y)$ iff

- $(Y, \leq_Y)$ is a subframe of $(X, \leq_X)$,
- $S_Y = S_X \cap Y$.

Equivalently:

Definition

$(Y, \leq_Y, R_Y)$ is a steady subframe of $(X, \leq_X, R_X)$ iff

- $(Y, \leq_Y)$ is a subframe of $(X, \leq_X)$,
- R_Y is the largest lax relation contained in $R_X \cap Y^2$.

Steady logics

Definition

A lax logic is steady iff it is generated by a class of lax frames closed under steady subframes.

Steady logics have the finite domain property for a slightly weaker class of morphisms.

These morphisms correspond to $\{\wedge, \to, \bot\}$-homomorphisms $h : A \to B$ between nuclear algebras such that

$$\bigcirc ha \leq h\bigcirc a$$

for all $a \in A$. Ergo, they preserve $\bigcirc$ in only one direction.

Steady canonical formulas

A steady canonical formula $\sigma(A, D_\vee, D_\circ)$ encodes $\bigcirc$ in the steady direction and the other direction only for $D_\circ$. The other connectives are encoded exactly the same as in the other canonical formulas.

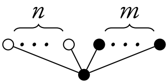
A steady formula is steady canonical formula of the form $\sigma(A) = \sigma(A, \emptyset, \emptyset)$.

Theorem

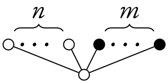
The following are equivalent

- L is steady.
- L is axiomatized by steady formulas.
- The class of L-algebras is closed under steady subalgebras.

Examples of steady logics



$\mathfrak{F}_{\bullet}^{n,m}$



$\mathfrak{F}_{\circ}^{n,m}$

$n + m \geq 2$

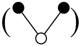
Theorem

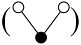
1. $\mathsf{PLL} \oplus \sigma(\mathfrak{F}_{\bullet}^{n,m})$ is the logic of all finite rooted frames that do not have $n + m$ maximal elements with at least n nuclear.
2. $\mathsf{PLL} \oplus \sigma(\mathfrak{F}_{\circ}^{n,m})$ is the logic of all finite $\circ$-rooted frames that do not have $n + m$ maximal elements with at least n nuclear.

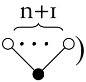
More examples of steady logics

LMx = PLL ⊕ $\sigma($$)$

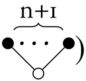
LRt = PLL ⊕ $\sigma($$)$

LIC = PLL ⊕ $\sigma($$)$

LLn = PLL ⊕ $\sigma($$)$

BIW_n = PLL ⊕ $\sigma($

$)$

BRW_n = PLL ⊕ $\sigma($

$)$

Lax Dummett-Lemmon conjecture

Now that we have obtained canonical formulas for the lax logic we can prove our conjecture:

Conjecture

If L is Kripke-complete then $\mathsf{L}^\bullet$ is Kripke-complete.

Steady canonical formulas characterize the structure of $\mathsf{L}^\bullet$ perfectly:

$$\mathsf{L} = \mathsf{IPC} \oplus \bigwedge_i \alpha(A_i, D_i) \iff \mathsf{L}^\bullet = \mathsf{PLL} \oplus \bigwedge_i \sigma(A_i, D_i, \emptyset).$$

Consequently, they are elegant machinery to prove preservation results involving the outer space translation!

Essentially, we want two prove two lemmas:

1. Find $\psi \notin \mathsf{L}$ for every $\phi \notin \mathsf{L}^\bullet$.
2. Extend an L-frame $X \nVdash \psi$ to an $\mathsf{L}^\bullet$-frame $X' \nVdash \phi$.

We can assume $\phi = \sigma(A, D_\vee, D_\circ)$. Then $\alpha(A, D_\vee)$ seems a good candidate for ψ.

Lemma 1.

If $\mathsf{L}^\bullet \nvdash \sigma(A, D_\vee, D_\circ)$ then $\mathsf{L} \nvdash \alpha(A, D_\vee)$.

$$B \twoheadrightarrow C \xhookleftarrow[D_\vee D_\circ]{h} A$$

We can simply drop $\bigcirc$.

Lemma 2.

Suppose X is an L-frame refuting $\alpha(A, D_\vee)$. Then there exists a lax relation on R such that (X, R) is an $\mathsf{L}^\bullet$-frame refuting $\sigma(A, D_\vee, D_\circ)$.

We can define the lax relation on X be finding the right subset!

Proof sketch. Let X^* denote the complex algebra of X, and A_* the dual space of A. By the refutation criterion:

$$X^* \twoheadrightarrow C \xleftarrow[D_\vee]{} A$$

Dually:

$$X \xhookrightarrow{h} (X^*)_* \xleftarrow{g} C_* \xrightarrow[D_\vee]{f} A_*$$

We can see A_* as an S-frame $(A_*, \leq, S_A)$. Define $S_X = h^{-1}[g[f^{-1}[S_A]]]$. ∎

Preservation results

Theorem

If L is Kripke-complete then $\mathsf{L}^\bullet$ is Kripke-complete.

We can prove similarly that:

Theorem

$\mathsf{L}^\bullet$ preserves:

- finite model property
- tabularity
- decidability (if Kripke-complete)

Conclusion

Lax canonical formulas and steady canonical formulas can be used to axiomatise all lax logics.

However, steady canonical formulas describe subtleties of the structure of lax logics in a clearer manner.

Besides, it seems unfeasible to generalize lax canonical formulas to other intuitionistic modal logics since they heavily rely on the local finiteness of the $\vee$-free reduct.

Steady canonical formulas on the other hand do not strictly make use of this reduct.

Future work

Axiomatize logics extending IK4 with “co-steady” canonical formulas.

Preservation results for less simple translations of intermediate logics into lax logics.

Investigating “semantic” translations.

Admissible rules for lax logics.

Thank you!